Advanced Methods Differential Equations Assignment 2

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Q1. Boundary layers

Let $0 < \varepsilon \ll 1$ and consider

$$\varepsilon y'' - y(y' + y) = 0\,, \quad 0 < x < 1\,, \quad \text{where} \quad y(0) = e\,,\ y(1) = 3\,. \tag{1.1}$$

Given that there is a boundary layer at $x = 1$, we want to find the outer, inner and uniformly valid expansion to leading order.

Since there is a boundary layer at $x = 1$, we may start by making a simple change of variables $z = 1 - x$, which gives $\frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = -\frac{d}{dz}$, so that we are now considering a boundary layer at $z = 0$, and (1.1) becomes (where $y = y(z)$)

$$\varepsilon y'' + yy' - y^2 = 0\,, \quad 0 < z < 1\,, \quad \text{where} \quad y(0) = 3\,, \quad y(1) = e\,. \tag{1.2}$$

We first consider the outer solution $y_{\text{out}}(z) = \sum_{n=0}^{\infty} \varepsilon^n y_n(z) = y_0 + \varepsilon y_1 + \dots$ in the outer region $\delta \ll z < 1$, so substituting into (1.1) this gives

$$\varepsilon(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)\big((y_0' - y_0) + \varepsilon(y_1' - y_1) + \varepsilon^2(y_2' - y_2) + \dots\big) = 0\,,$$

but since we are in the outer region, the leading order term will dominate, so we have $y_{\text{out}}(z) \approx y_0(z)$, so by comparing orders we have

$$O(1): \quad y_0(y_0' - y_0) = 0\,, \quad \text{so} \quad y_0 = 0 \ \text{ or } \ y_0' - y_0 = 0\,.$$

The first solution is trivial and gives no boundary layer, meaning we must be in the situation of $y_0' = y_0$, so $y_0 = Ae^z$. Using $y(1) = e$ away from the boundary layer, this gives

$$y_{\text{out}}(z) = y_0(z) = e^z = e^{1-x}\,. \tag{1.3}$$

For the inner solution, we start by stretching the region to $z = \delta Z$, so $\frac{d}{dz} = \frac{1}{\delta}\frac{d}{dZ}$, which turns our equation (1.2) into (where $y_{\text{in}}(z) = Y_{\text{in}}(Z)$),

$$\frac{\varepsilon}{\delta^2} Y_{\text{in}}'' + \frac{1}{\delta} Y_{\text{in}} Y_{\text{in}}' - Y_{\text{in}}^2 = 0 \quad \text{for} \quad \delta \to 0\,. \tag{1.4}$$

We can then apply a dominant balance argument: first suppose $\delta \ll \varepsilon$, so $\frac{\varepsilon}{\delta^2} \gg \frac{1}{\delta} \gg 1$, which gives $Y_{\text{in}}'' = 0$ so $Y_{\text{in}}(Z) = AZ + B$, but this diverges as $Z \to \infty$ so it couldn't be matched. If $\delta \gg \varepsilon$, so $\frac{\varepsilon}{\delta} \ll \frac{1}{\delta} \ll 1$, this would give the $\frac{1}{\delta} Y_{\text{in}} Y_{\text{in}}'$ term dominating, giving $Y_{\text{in}} = 0$ or $Y_{\text{in}} = Z$, both of which cannot be matched. Therefore we must have $\delta = \varepsilon$ and so (1.4) becomes

$$\frac{1}{\varepsilon} Y_{\text{in}}'' + \frac{1}{\varepsilon} Y_{\text{in}} Y_{\text{in}}' - Y_{\text{in}}^2 = 0\,. \tag{1.5}$$

Letting $Y_{\text{in}}(Z) = \sum_{n=0}^{\infty} \varepsilon^n Y_n = Y_0 + \varepsilon Y_1 + \dots$, we have

$$\frac{1}{\varepsilon}(Y_0'' + \varepsilon Y_1'' + \varepsilon^2 Y_2'' + \dots) + \frac{1}{\varepsilon}(Y_0' + \varepsilon Y_1' + \varepsilon^2 Y_2' + \dots)(Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots)$$
$$-(Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots)^2 = 0\,. \quad (1.6)$$

We only need to consider the $O(\frac{1}{\varepsilon})$ term in the leading order as $\varepsilon \to 0$, so we have

$$O\left(\frac{1}{\varepsilon}\right): \quad Y_0'' + Y_0' Y_0 = 0\,. \quad (1.7)$$

To solve the $O(\frac{1}{\varepsilon})$ equation, we note the identity $\frac{d}{dx} y(x)^2 = 2y'y$, so integrating both sides we have

$$\int (Y_0'' + Y_0' Y_0) dZ = Y_0' + \frac{1}{2} Y_0^2 - C = 0\,,$$

$$\text{so} \quad \int \frac{1}{C - \frac{1}{2} Y_0^2} dY_0 = \int dZ\,, \quad \text{so} \quad \frac{\sqrt{2}}{\sqrt{C}} \text{arctanh}\left(\frac{Y_0}{\sqrt{2C}}\right) = Z + A\,.$$

Letting $B = \sqrt{2C}$ we can rearrange this to get

$$Y_0(Z) = B \tanh\left(\frac{B}{2}(Z + A)\right) \quad (1.8)$$

for some constants A and B. We can then apply the boundary condition at the boundary layer (which must be valid for the highest order term), $y(0) = 3$, to see

$$3 = B \tanh\left(\frac{AB}{2}\right) = B \frac{e^{AB} - 1}{e^{AB} + 1}\,,$$

$$\text{so} \quad (3 - B) e^{AB} + (3 + B) = 0\,, \quad \text{so} \quad A = \frac{1}{B} \log\left(\frac{B+3}{B-3}\right)\,.$$

Using the identity $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$, we can thus rewrite (1.8) as

$$Y_0(Z) = B \frac{\tanh\left(\frac{B}{2} Z\right) + \tanh\left(\frac{1}{2} \log\left(\frac{B+3}{B-3}\right)\right)}{1 + \tanh\left(\frac{B}{2} Z\right) \tanh\left(\frac{1}{2} \log\left(\frac{B+3}{B-3}\right)\right)} = \frac{B^2 \tanh\left(\frac{B}{2} Z\right) + 3B}{B + 3 \tanh\left(\frac{B}{2} Z\right)}\,, \quad (1.9)$$

where in the second equality we used the following simple calculation:

$$\tanh\left(\frac{1}{2} \log\left(\frac{B+3}{B-3}\right)\right) = \frac{\frac{B+3}{B-3} - 1}{\frac{B+3}{B-3} + 1} = \frac{B + 3 - B + 3}{B + 3 + B - 3} = \frac{3}{B}\,.$$

To determine B we want to use the matching condition $\lim_{Z\to\infty} Y_{\text{in}}(Z) = \lim_{z\to 0} y_{\text{out}}$. Noting that $\lim_{x\to\infty} \tanh(kx) = \text{sign}(k)$ (i.e. $+1$ if $k > 0$ and -1 if $k < 0$) we have

$$\lim_{Z\to\infty} Y_{\text{in}}(Z) = \frac{B^2 \text{sign}(B) + 3B}{B + 3\text{sign}(B)} = 1 = \lim_{z\to 0} y_{\text{out}}\,, \quad \text{so} \ B = \pm 1\,. \quad (1.10)$$

Either option will give the same solution so we can take $B = 1$ for simplicity. Therefore,

$$Y_{\text{in}}(Z) = \frac{\tanh\left(\frac{1}{2} Z\right) + 3}{3 \tanh\left(\frac{1}{2} Z\right) + 1} = \frac{2e^Z + 1}{2e^Z - 1} = \frac{2e^{\frac{z}{\varepsilon}} + 1}{2e^{\frac{z}{\varepsilon}} - 1}\,. \quad (1.11)$$

Using the fact that $y_{\text{match}} = \lim_{z\to 0} y_{\text{out}} = 1$ and recalling that $z = 1 - x$, we finally have

$$y_{\text{unif}}(x) = y_{\text{out}}(x) + y_{\text{in}}(x) - y_{\text{match}} = e^{1-x} + \frac{2e^{\frac{1-x}{\varepsilon}} + 1}{2e^{\frac{1-x}{\varepsilon}} - 1} - 1\,. \quad (1.12)$$

It is easily verified that this satisfies the desired properties and so we are done. □

Q2. Internal boundary layer

Consider

$$\varepsilon y'' + (x^2 - \frac{1}{4})y' = 0\,, \quad 0 < x < 1\,, \quad \text{where} \quad y(0) = 1,\ \ y(1) = -1\,. \tag{2.1}$$

Part a)

Denoting $a(x) = x^2 - \frac{1}{4} = (x - \frac{1}{2})(x + \frac{1}{2})$, we see that $a(\frac{1}{2}) = 0$ (note that $-\frac{1}{2} \notin (0,1)$), meaning that there is a singularity of the ODE at $x = \frac{1}{2}$. In such a region $a(x) \sim O(\varepsilon)$ meaning there can be rapid changes in y'', hence meaning we must go through a boundary layer at $x = \frac{1}{2}$ by the remarks in W7 (page 6) of the lecture notes.

Part b)

First consider the region $x < \frac{1}{2}$, where we set $y_{\text{out}}(x) = y_0(x) + \varepsilon y_1(x) + \dots$, then we have

$$\varepsilon\big(y_0'' + \varepsilon y_1'' + \dots\big) + a(x)\big(y_0' + \varepsilon y_1' + \dots\big) = 0\,, \tag{2.2}$$

so to leading order (i.e. analysing the $O(1)$ terms) we see that

$$a(x)y_0' = 0\,, \quad \text{so} \quad y_0(x) = C_-\,,$$

for some constant C_-, and so applying $y(0) = 1$ we have $y_0(x) = 1$. Since this is an outer solution, we only consider $O(1)$ terms as $O(\varepsilon)$ is very small in this region, so we have $y_{\text{out}}(x) = 1$ for $x < \frac{1}{2}$.

Performing an identical analysis with the same expansion as in (2.2), shows that for $x > \frac{1}{2}$ we must have $y_0(x) = C_+$ for some constant C_+, hence applying $y(1) = -1$ we have $y_0(x) = -1$, so $y_{\text{out}}(x) = -1$ for $x > \frac{1}{2}$.

Part c)

To determine the inner solution about $x = \frac{1}{2}$ we will make a change of variables $z = x - \frac{1}{2}$ to simplify our analysis to have a boundary layer $z = 0$, still in the interior of the domain. Noting that $\frac{d}{dx} = \frac{d}{dz}$, (2.1) becomes

$$\varepsilon y'' + z(z+1)y' = 0\,, \quad -\frac{1}{2} < z < \frac{1}{2}\,, \quad \text{where} \quad y(-\frac{1}{2}) = 1,\ \ y(\frac{1}{2}) = -1\,, \tag{2.3}$$

where we denote $a(z) = z(z+1)$. Now let $z = \delta Z$ (so $\frac{d}{dz} = \frac{1}{\delta}\frac{d}{dZ}$ and $y_{\text{in}}(z) = Y_{\text{in}}(Z)$), then (2.3) becomes

$$\frac{\varepsilon}{\delta^2}Y_{\text{in}}'' + \frac{a(\delta Z)}{\delta}Y_{\text{in}}' = 0\,.$$

Since we are near a boundary layer, we may write $a(z) \approx a'(0)z$ as $z \to 0$ and calculate $a'(z) = 2z + 1$ so $a'(0) = 1$, so $a(\delta Z) \approx \delta Z$ and our equation becomes

$$\frac{\varepsilon}{\delta^2}Y_{\text{in}}'' + ZY_{\text{in}}' = 0\,. \tag{2.4}$$

We may then perform a dominant balance. First suppose $\delta \ll \varepsilon$ which implies $\frac{\varepsilon}{\delta^2} \gg \frac{1}{\delta} \gg 1$ which gives a dominant Y_{in}'' term, so $Y_{\text{in}}(Z) = AZ + B$ for some constant A and B. But then $\lim_{Z\to\infty} Y_{\text{in}} = \infty$, so we couldn't match and so this can't be the balance. Alternatively, if $\delta \gg \varepsilon$, then $\frac{\varepsilon}{\delta} \ll 1$, meaning the ZY_{in}' term dominates and so $Y_{\text{in}} = A$ for some constant A.

But then we again cannot match the inner and outer solutions at $Z \to \infty$ unless $A = \pm 1$ (depending on the region), at which point there would be no boundary layer. Thus we must have $\frac{\varepsilon}{\delta^2} \sim 1$, so our equation becomes

$$Y_{\text{in}}'' + Z Y_{\text{in}}' = 0\,. \tag{2.5}$$

We can then solve this by introducing the integrating factor of $I = \exp\left(\int Z dZ\right) = \exp\left(\frac{1}{2}Z^2\right)$, so

$$\frac{d}{dZ}(e^{\frac{1}{2}Z^2} Y_{\text{in}}') = 0\,, \quad \text{so} \quad Y_{\text{in}}'(Z) = C e^{-\frac{1}{2}Z^2}\,, \quad \text{so} \quad Y_{\text{in}}(Z) = C \int_0^Z e^{-\frac{1}{2}t^2} dt\,. \tag{2.6}$$

To solve for C we need to impose the matching condition $\lim_{Z\to\pm\infty} Y_{\text{in}}(Z) = \lim_{z\to 0^\pm} y_{\text{out}}$, but this will be different in the different regions. We note the identity $\int_0^\infty e^{-\frac{1}{2}t^2} dt = \sqrt{\frac{\pi}{2}}$. Then for $z < 0$ (i.e. $x < \frac{1}{2}$) where $y_{\text{out}}(z) = 1$ we solve $\lim_{Z\to-\infty} Y_{\text{in}}(Z) = \lim_{z\to 0^+} y_{\text{out}}(z)$, so

$$C_- \int_0^{-\infty} e^{-\frac{1}{2}t^2} dt = 1\,, \quad \text{so} \quad Y_{\text{in}}(Z) = -\sqrt{\frac{2}{\pi}} \int_0^Z e^{-\frac{1}{2}t^2} dt \quad \text{for} \;\; Z < 0\,. \tag{2.7}$$

Similarly, for $z > 0$ we have $y_{\text{out}}(z) = -1$ so $C_+ = -\sqrt{\frac{2}{\pi}}$ and so

$$Y_{\text{in}}(Z) = -\sqrt{\frac{2}{\pi}} \int_0^Z e^{-\frac{1}{2}t^2} dt \quad \text{for} \;\; Z > 0\,. \tag{2.8}$$

Part d)

To find the uniformly valid solution we define $y_{\text{match}}(z) = \lim_{z\to 0} y_{\text{out}}(z)$, which in both cases gives us $y_{\text{match}}(z) = y_{\text{out}}$ since y_{out} is a constant. We see that in writing $y_{\text{unif}} = y_{\text{in}} + y_{\text{out}} - y_{\text{match}} = y_{\text{in}}$, and noting that $Y_{\text{in}}(Z)$ is the same in both cases from (2.7) and (2.8), for all $z \in \mathbb{R}$ (i.e. all $x \in \mathbb{R}$) we have a uniformly valid expansion to leading order of

$$y_{\text{unif}}(Z) = -\sqrt{\frac{2}{\pi}} \int_0^Z e^{-\frac{1}{2}t^2} dt = -\sqrt{\frac{2}{\pi}} \int_0^{\frac{x-\frac{1}{2}}{\varepsilon}} e^{-\frac{1}{2}t^2} dt = y_{\text{unif}}(x)\,. \tag{2.9}$$

We note that this is, up to rescaling, the so-called error function (Gaussian CDF), which for small ε will be very steep around the boundary layer $x = \frac{1}{2}$. $\square$

Q3. WKB analysis

Consider

$$\varepsilon^2 y'' + (1+x)^4 y = 0\,, \quad \text{for } x > 0\,. \tag{3.1}$$

We want to perform WKB analysis on this equation.

Part a)

We first note that in writing $Q(x) = -(1+x)^4$ we have Schrödinger's equation $\varepsilon^2 y'' = Q(x)y$. We start by assuming y has the form

$$\begin{aligned}
y &\sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right]\,,\\
\text{so}\quad y' &\sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)\exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right]\,,\\
\text{so}\quad y'' &\sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x) + \frac{1}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n(x)\right)^2\right)\exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right]\,.
\end{aligned} \tag{3.2}$$

Using the Cauchy product expansion we can write the coefficient of the exponential in y'' as

$$\frac{1}{\delta^2}S_0'^2 + \frac{1}{\delta}(2S_0'S_1' + S_0'') + (S_1'' + S_1'^2 + 2S_0'S_2') + O(\delta)\,.$$

So, substituting these equations into (3.1) and dividing by the exponential, we have

$$\frac{\varepsilon^2}{\delta^2}S_0'^2 + \frac{\varepsilon^2}{\delta}(2S_0'S_1' + S_0'') + \varepsilon^2(S_1'' + S_1'^2 + 2S_0'S_2') + \varepsilon^2 O(\delta) = Q(x)\,. \tag{3.3}$$

We may then perform a dominant balance analysis to determine $\delta(\varepsilon)$. Let T1, T2 and T3 denote the terms associated to $\frac{\varepsilon^2}{\delta}$, $\frac{\varepsilon^2}{\delta}$ and 1 (i.e. $Q(x)$) respectively (we can safely ignore $O(\varepsilon^2)$ terms as $\varepsilon \to 0$). First assume that T1 $\ll$ T2 $\sim$ T3, so $\delta = \varepsilon^2$, giving $\frac{1}{\varepsilon^2}S_0'^2 \ll 2S_0'S_1' + S_0'' \sim Q(x)$. But then as $\varepsilon \to 0$ the left hand side of this will go to ∞, which contradicts the fact that it is much less than $Q(x)$ which does not diverge, thus giving a contradiction. If we then suppose T3 $\ll$ T1 $\sim$ T2, this would imply $\frac{\varepsilon^2}{\delta^2} = \frac{\varepsilon^2}{\delta}$, so $\delta = 1$. But then all terms on the left hand side of (3.3) go to 0 as $\varepsilon \to 0$, which contradicts $Q(x) \ll \mathrm{T1}, \mathrm{T2}$ hence we have another contradiction.

Therefore, dominant balance tells us that $\frac{\varepsilon^2}{\delta^2}$ must have the same order of magnitude as $Q(x)$, so δ is proportional to ε so we may just take $\delta = \varepsilon$. We then have the first few orders as

$$\begin{aligned}
O(1):&\quad S_0'^2 = -(1+x)^4\,,\\
O(\varepsilon):&\quad 2S_0'S_1' + S_0'' = 0\,,\\
O(\varepsilon^2):&\quad S_1'' + S_1'^2 + 2S_0'S_2' = 0\,.
\end{aligned} \tag{3.4}$$

Hence we can solve $S_0' = \pm i(1+x)^2$, so

$$S_0(x) = \int \pm i(1+x)^2 dx = \pm\frac{i}{3}(x+1)^3 + C_\pm\,. \tag{3.5}$$

The leading order solution is considered to be all non-negligible terms in the limit $\varepsilon \to 0$, meaning we want to solve the $O(\varepsilon)$ equation as well. Since $S_0' = \pm i(1+x)^2$ from before, meaning $S_0'' = \pm 2i(1+x)$, we have (noting that $x+1>0$ so $\log|x+1| = \log(x+1)$),

$$2S_0'S_1' + S_0'' = \pm 2i(1+x)^2 S_1' \pm 2i(1+x) = 0\,,$$
$$\text{so} \quad S_1 = \int -\frac{1}{x+1}dx = -\log(x+1) + D_\pm\,.$$

Noting that our two possible solutions for $S_0(x)$ are linearly independent solutions (giving us a sum of exponentials in the final solution) and writing $C_1 = \exp(\frac{1}{\varepsilon}C_+ + D_+)$ and $C_2 = \exp(\frac{1}{\varepsilon}C_- + D_-)$, we have the leading order solution

$$y(x) \sim \frac{C_1}{x+1}\exp\left[\frac{i}{3\varepsilon}(1+x)^3\right] + \frac{C_2}{x+1}\exp\left[-\frac{i}{3\varepsilon}(1+x)^3\right]\,. \tag{3.6}$$

We note that the presence of the i in the exponential will give periodic solutions (ultimately due to the fact that $Q(x) < 0$ for all x), but it is more convenient to leave it in exponential form for the moment.

Part b)

We can then impose the boundary conditions $y(0) = 0$ and $y'(0) = 1$. The first one gives us

$$0 = C_1 e^{\frac{i}{3\varepsilon}} + C_2 e^{-\frac{i}{3\varepsilon}}\,. \tag{3.7}$$

For $f(x) = \frac{A}{x+1}\exp\left[k(1+x)^3\right]$ where k and A are some constants, we have

$$f'(x) = \frac{A}{(x+1)^2}\left(3k(x+1)^3 - 1\right)e^{k(1+x)^3}\,,$$
$$\text{so} \quad y'(x) = \frac{C_1}{(x+1)^2}\left(\frac{i}{\varepsilon}(x+1)^3 - 1\right)e^{\frac{i}{3\varepsilon}(1+x)^3} - \frac{C_2}{(x+1)^2}\left(\frac{i}{\varepsilon}(x+1)^3 + 1\right)e^{-\frac{i}{3\varepsilon}(1+x)^3}\,.$$

Hence applying our second condition we have

$$1 = C_1\left(\frac{i}{\varepsilon} - 1\right)e^{\frac{i}{3\varepsilon}} - C_2\left(\frac{i}{\varepsilon} + 1\right)e^{-\frac{i}{3\varepsilon}} = C_1\left(\frac{i}{\varepsilon} - 1\right)e^{\frac{i}{3\varepsilon}} + C_1\left(\frac{i}{\varepsilon} + 1\right)e^{\frac{i}{3\varepsilon}} = \frac{2C_1 i}{\varepsilon}e^{\frac{i}{3\varepsilon}}\,,$$

where we used (3.7) in the second equality. Rearranging we find that

$$C_1 = \frac{\varepsilon}{2i}e^{-\frac{i}{3\varepsilon}}\,, \quad \text{so} \quad C_2 = -\frac{\varepsilon}{2i}e^{\frac{i}{3\varepsilon}}\,, \tag{3.8}$$

which gives a leading order solution of

$$y(x) \sim \frac{\varepsilon}{2i(x+1)}e^{\frac{i}{3\varepsilon}\left((x+1)^3-1\right)} - \frac{\varepsilon}{2i(x+1)}e^{-\frac{i}{3\varepsilon}\left((x+1)^3-1\right)} = \frac{\varepsilon}{i(x+1)}\sinh\left(\frac{i}{3\varepsilon}\left((x+1)^3 - 1\right)\right)\,,$$

which, using the fact that $\sinh(ix) = i\sin(x)$, finally simplifies to

$$y(x) \sim \frac{\varepsilon}{(x+1)}\sin\left(\frac{(x+1)^3 - 1}{3\varepsilon}\right)\,. \tag{3.9}$$

Part c)

To determine the region of validity of the WKB approximation, we first want to solve for S_2 in the $O(\varepsilon^2)$ equation of (3.4), which gives

$$0 = S_1'' + S_1'^2 + 2S_0'S_2' = \frac{1}{(x+1)^2} + \frac{1}{(x+1)^2} \pm 2i(x+1)^2 S_2' \,,$$
$$\text{so} \quad S_2' = \mp\frac{1}{i}(x+1)^{-4}\,, \quad \text{so} \quad S_2 = \pm\frac{1}{3i}\frac{1}{(x+1)^3} + E_\pm\,. \tag{3.10}$$

We know from lectures that our leading order WKB approximation is valid on some interval $I \subseteq \mathbb{R}$ if the following two conditions are met (where $\delta = \varepsilon$):

$$\varepsilon S_2 \ll S_1 \ll \frac{1}{\varepsilon}S_0\,, \quad \text{and} \quad \varepsilon S_2 \ll 1\,, \quad \text{as } \varepsilon \to 0\,. \tag{3.11}$$

When performing such asymptotic calculations we may discard coefficients of S_i terms (we only care about the x behaviour) and arbitrary constants $C_\pm, D_\pm, E_\pm$ as they are also negligible in the asymptotic expansions. Thus our first condition is

$$\varepsilon\frac{1}{(x+1)^3} \approx \varepsilon S_2 \ll S_1 \approx \log(x+1)\,,$$

and so letting $x+1 = \varepsilon^\alpha$ for some $\alpha \in \mathbb{R}$ we require

$$\varepsilon \ll \alpha\varepsilon^{3\alpha}\log\varepsilon\,, \quad \text{so} \quad 1 \ll \alpha\varepsilon^{3\alpha-1}\log\varepsilon\,, \quad \text{so} \quad 3\alpha - 1 < 0\,, \quad \text{so} \quad \alpha < \frac{1}{3} \tag{3.12}$$

meaning our first requirement is $x+1 \gg \varepsilon^{\frac{1}{3}}$. Note that the conclusion that $3\alpha - 1 < 0$ follows from the requirement that $\alpha\varepsilon^{3\alpha-1}\log\varepsilon$ be much greater than 1 for small ε. Next we have

$$\log(x+1) \approx S_1 \ll \frac{1}{\varepsilon}S_0 \approx \frac{1}{\varepsilon}(x+1)^3\,, \tag{3.13}$$

so again taking $x+1 = \varepsilon^\alpha$ this gives

$$1 \ll \frac{\varepsilon^{3\alpha-1}}{\alpha\log\varepsilon} \tag{3.14}$$

which is true for any value of α. Our final condition gives

$$\varepsilon\frac{1}{(x+1)^3} \ll 1\,, \quad \text{so } x+1 \gg \varepsilon^{\frac{1}{3}}\,, \tag{3.15}$$

which we note is the same as the first condition above. Therefore the WKB leading order approximation is valid for $x+1 \gg O(\varepsilon^{1/3})$.

Q4. Multiple time scales

Consider $y(t)$ satisfying the equation

$$\ddot{y} + y + \varepsilon y \dot{y}^2 = 0\,, \quad t > 0\,, \quad y(0) = 0\,,\ \dot{y}(0) = 1\,. \tag{4.1}$$

We want to use the method of multiple time scales, with $T_0 = t$ and $T_1 = \tau = \varepsilon t$ to determine the leading order term of the uniformly valid asymptotic expansion of $y(t)$.

We begin by assuming

$$y(t) = Y(t,\tau) = \sum_{n=0}^{\infty} \varepsilon^n Y_n(t,\tau) = Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \varepsilon^2 Y_2(t,\tau) + \dots$$
$$\text{with} \quad Y(0,0) = 0\,, \quad \text{and} \quad \left.\frac{\partial Y_0}{\partial t}\right|_{(0,0)} = 1\,, \tag{4.2}$$

which gives derivatives of

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial t} + \varepsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right) + O(\varepsilon^2)\,, \tag{4.3}$$
$$\text{and} \quad \frac{d^2 y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial t \partial \tau} + \frac{\partial^2 Y_1}{\partial t^2}\right) + O(\varepsilon^2)\,.$$

Substituting these into (4.1), we have (neglecting higher order terms since we are only interested in the leading order)

$$0 = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial t\partial \tau} + \frac{\partial^2 Y_1}{\partial t^2}\right) + Y_0 + \varepsilon Y_1 + \varepsilon\left(Y_0 + \varepsilon Y_1\right)\left(\frac{\partial Y_0}{\partial t} + \varepsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right)\right)^2 + O(\varepsilon^2)$$
$$= \left(\frac{\partial^2 Y_0}{\partial t^2} + Y_0\right) + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial t \partial \tau} + \frac{\partial^2 Y_1}{\partial t^2} + Y_1 + Y_0\left(\frac{\partial Y_0}{\partial t}\right)^2\right) + O(\varepsilon^2)\,. \tag{4.4}$$

Thus our $O(1)$ equation is

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0\,, \quad \text{so } Y_0 = A(\tau)e^{it} + \overline{A}(\tau)e^{-it} \tag{4.5}$$

for some function $A = A(\tau)$ where $\overline{A}$ denotes the conjugate, since Y_0 is a real function. Then, our $O(\varepsilon)$ equation is

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\partial\tau} - Y_0\left(\frac{\partial Y_0}{\partial t}\right)^2$$
$$= -2\left(A'(\tau)ie^{it} - \overline{A}'(\tau)ie^{-it}\right) - \left(A(\tau)e^{it} + \overline{A}(\tau)e^{-it}\right)\left(i\left(A(\tau)e^{it} - \overline{A}(\tau)e^{-it}\right)\right)^2$$
$$= \left(-2iA' - A^2\overline{A}\right)e^{it} + \left(2i\overline{A}' - A\overline{A}^2\right)e^{-it} + A^3 e^{3it} + \overline{A}^3 e^{-3it}\,. \tag{4.6}$$

The homogeneous solution of this equation is

$$Y_{1,\text{hom}}(t) = B(\tau)e^{it} + \overline{B}(\tau)e^{-it}\,, \tag{4.7}$$

which has a frequency of 1, which suggests that the $e^{\pm it}$ terms in (4.6) will cause secular solutions. Thus, to avoid secular solutions we require $A(\tau)$ to be such that

$$2iA'(\tau) + A^2(\tau)\overline{A}(\tau) = 0\,, \quad \text{and} \quad 2i\overline{A}'(\tau) - A(\tau)\overline{A}^2(\tau) = 0\,, \tag{4.8}$$

where the second equation is the complex conjugate of the first so we just require a solution to the first equation. To do this we apply a separation of variables technique (in some sense) and let

$$A(\tau) = R(\tau)e^{i\Theta(\tau)}\,, \quad \text{so} \quad \frac{dA}{d\tau} = (R' + iR\Theta')e^{i\Theta}\,, \tag{4.9}$$

for some real functions R and Θ. Substituting this into the above we have

$$2i(R' + iR\Theta')e^{i\Theta} + (R^2 e^{2i\Theta})(Re^{-i\Theta}) = (2iR' - 2R\Theta' + R^3)e^{i\Theta} = 0\,.$$

After dividing by $e^{i\Theta}$, the real part of the equation gives

$$-2R\Theta' + R^3 = 0\,, \quad \text{so} \quad \Theta'(\tau) = \frac{1}{2}R^2\,,$$

and the imaginary part gives $2iR'(\tau) = 0$, so

$$R(\tau) = R(0)\,, \quad \text{and} \quad \Theta(\tau) = \frac{1}{2}R(0)^2\tau + \Theta(0)\,,$$

so we finally have

$$A(\tau) = R(0)e^{i\left(\frac{1}{2}R(0)^2\tau + \Theta(0)\right)}\,. \tag{4.10}$$

We can hence write Y_0 as

$$\begin{aligned} Y_0(t) &= R(0)e^{i\left(\frac{1}{2}R(0)^2\tau + \Theta(0) + t\right)} + R(0)e^{-i\left(\frac{1}{2}R(0)^2\tau + \Theta(0) + t\right)} \\ &= 2R(0)\cos\left(\frac{1}{2}R(0)^2\tau + \Theta(0) + t\right)\,. \end{aligned} \tag{4.11}$$

Applying our boundary conditions in (4.2) we have

$$Y_0(0,0) = 2R(0)\cos\big(\Theta(0)\big) = 0\,, \quad \text{and} \quad \left.\frac{\partial Y_0}{\partial t}\right|_{(0,0)} = -2R(0)\sin\big(\Theta(0)\big) = 1\,,$$

which thus gives (noting that the non-uniqueness of $\Theta(0)$ is ultimately not problematic as it has the same effect in (4.13))

$$\Theta(0) = \frac{\pi}{2}\,, \quad \text{and} \quad R(0) = -\frac{1}{2}\,. \tag{4.12}$$

Plugging this into (4.11), using $\tau = \varepsilon t$ and the fact that $\cos(x + \pi/2) = -\sin(x)$, we have

$$Y_0(t) = -\cos\left(\frac{1}{8}\tau + \frac{\pi}{2} + t\right) = \sin\left(\left(1 + \frac{1}{8}\varepsilon\right)t\right)\,. \tag{4.13}$$

Therefore our leading order solution is

$$y(t) = \sin\left(\left(1 + \frac{1}{8}\varepsilon\right)t\right) + O(\varepsilon)\,, \quad \text{as } \varepsilon \to 0^+\,,\ \varepsilon t = O(1)\,. \tag{4.14}$$

We see that the period is $T \sim \frac{1}{1+\frac{1}{8}\varepsilon} \sim 1 - \frac{1}{8}\varepsilon$ where the second equality uses the Taylor expansion of $\frac{1}{1+\varepsilon}$ for small ε. □