Algebraic Geometry Assignment 3

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1 Category Theory

Q8. Subfunctors

Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \mathsf{Sets}$ be a functor. $G \subseteq F$ is a subfunctor if for each $c \in \mathrm{Obj}(\mathcal{C})$ there is a subset $G(c) \subseteq F(c)$ such that for every $\phi : c \to d$ in $\mathcal{C}$, the restriction of the induced morphism $F(\phi) : F(c) \to F(d)$ to $G(c)$, factors through $G(d)$. That is, we have a diagram (where ι is the natural inclusion)

$$G(c) \xrightarrow{G(\phi)} G(d) \xrightarrow{\iota} F(d)\,, \qquad F(\phi)\big|_{G(c)} : G(c) \to F(d) \tag{1.8.1}$$

Part a)

Define $G : \mathcal{C} \to \mathsf{Sets} : c \mapsto G(c)$ satisfying the properties of a subfunctor of F above, we will show that this is a well defined functor. Indeed, $G(c)$ is an object of Sets since $G(c) \subseteq F(c)$ and $F(c)$ is an object of Sets. Let $\phi : c \to d$ and $\psi : d \to e$ be arrows in $\mathcal{C}$, then (dropping the cumbersome ι notation)

$$\begin{aligned} G(\psi) \circ G(\phi) &= F(\psi)\big|_{G(d)} \circ F(\phi)\big|_{G(c)} \\ &= F(\psi \circ \phi)\big|_{G(c)} \\ &= G(\psi \circ \phi)\,. \end{aligned} \tag{1.8.2}$$

Therefore G is a well defined functor.

Part b)

Let A be a ring and $f \in A$, we will show that $D_A(f) : \mathsf{Alg}(A) \to \mathsf{Sets}$ is a subfunctor of $\overline{A} : \mathsf{Alg}(A) \to \mathsf{Sets}$, where, denoting $(B, \beta) \in \mathrm{Obj}(\mathsf{Alg}(A))$ for a ring B and ring homomorphism $\beta : A \to B$, we have

$$D_A(f)((B,\beta)) = \begin{cases} \{\beta\} & \beta(f) \in B^\times \\ \varnothing & \text{otherwise} \end{cases}\,, \quad \text{and} \quad \overline{A}((B,\beta)) = \mathrm{Hom}_{\mathsf{Alg}(A)}(A, (B,\beta))\,, \tag{1.8.3}$$

where A is viewed as an A-algebra over itself.

To define how $D_A(f)$ acts on an A-algebra homomorphism $\psi : (B, \beta) \to (C, \gamma)$, we recall that this map is equivalent to a ring homomorphism $\psi : B \to C$ such that the following diagram commutes

$$\begin{array}{ccc} & A & \\ {}^{\beta}\swarrow & & \searrow^{\gamma} \\ B & \xrightarrow{\psi} & C \end{array} \quad . \tag{1.8.4}$$

Therefore we must have $\beta(f) \mapsto \psi(\beta(f)) = \gamma(f) \in C$. But since these are all ring homomorphisms, we know that if $\beta(f)$ is a unit in B then $\gamma(f)$ is also a unit in C. Hence, we can define how $D_A(f)$ acts on a morphism ψ as

$$D_A(f)(\psi) : D_A(f)((B,\beta)) \longrightarrow D_A(f)((C,\gamma)),$$
$$\begin{cases} \{\beta\} \overset{\psi\circ -}{\longmapsto} \{\gamma\} & \text{if } \beta(f) \in B^\times \\ \varnothing \longmapsto \varnothing & \text{otherwise} \end{cases} \tag{1.8.5}$$

which is well defined by our unit argument above. Then, given $\psi : (B,\beta) \to (C,\gamma)$ and $\phi : (C,\gamma) \to (D,\delta)$ such that we have

$$\begin{array}{ccccc} & & A & & \\ & {}^{\beta}\swarrow & \downarrow{\gamma} & \searrow^{\delta} & \\ B & \xrightarrow{\psi} & C & \xrightarrow{\phi} & D \end{array} \quad , \tag{1.8.6}$$

we use the same unit argument to see that

$$\Big(D_A(f)(\phi) \circ D_A(f)(\psi)\Big)(\{\beta\}) = D_A(f)(\phi)(\{\gamma\}) = \{\delta\} = D_A(f)(\phi \circ \psi)(\{\beta\}), \tag{1.8.7}$$

and similarly when applied to $\varnothing$, hence our composition holds and so $D_A(f)$ is a well defined functor. Note too that $\overline{A} : \mathsf{Alg}(A) \to \mathsf{Sets}$ acts on a morphism $\rho : (B,\beta) \to (C,\gamma)$ as $\overline{A}(\rho) : \overline{A}((B,\beta)) \to \overline{A}((C,\gamma)) : \lambda \mapsto \rho \circ \lambda$.

We then see that with our definition in (1.8.3), and noting that $\beta : A \to B$ can be viewed as an A-algebra homomorphism, we have $D_A(f)((B,\beta)) \subseteq \mathrm{Hom}_{\mathsf{Alg}(A)}(A,(B,\beta))$.

Finally, we see that in considering $\overline{A}(\psi)\Big|_{D_A(f)((B,\beta))}$, we have

$$\overline{A}(\psi)(\{\beta\}) = \{\psi \circ \beta\} = \{\gamma\} \quad \text{and} \quad \overline{A}(\psi)(\varnothing) = \varnothing, \tag{1.8.8}$$

so $\overline{A}(\psi)\Big|_{D_A(f)((B,\beta))}$ is clearly in the image of $D_A(f)((C,\gamma))$, hence the factoring through property holds. Therefore, $D_A(f)$ is a subfunctor of $\overline{A}$! $\square$

Q15. Localisation of a colimit

Let A be a ring and $S \subseteq A$ a multiplicative set. We endow S with the partial order: $s \leq t$ if $\frac{s}{1} \in (A_t)^\times$. In other words, $s \leq t$ if $t = us$ for some $u \in S$.

Part a)

A set S is directed if for any $s \in S$ and $s' \in S$ there is some $c \in S$ such that $s \leq c$ and $s' \leq c$. Indeed, since S is a multiplicative set, we can just take $c = ss'$, and by our definition of partial ordering we clearly have $s \leq c = ss'$ and $s' \leq c = ss'$. Hence S is directed with respect to the partial order.

Part b)

Suppose $s \leq t$, where $t = us$ for some $u \in S$, we can define an A-algebra homomorphism

$$\mu_{st} : A_s \to A_t \,, \qquad \text{where } \mu_{st}\left(\frac{a}{s^n}\right) = \frac{au^n}{t^n} \,. \tag{1.15.1}$$

We see this is a ring homomorphism since we have

$$\mu_{st}\left(\frac{a}{s^n} + \frac{b}{s^m}\right) = \mu_{st}\left(\frac{as^m + bs^n}{s^{n+m}}\right) = \frac{(as^m + bs^n)u^{n+m}}{t^{n+m}} = \frac{au^n}{t^n} + \frac{bu^m}{t^m} = \mu_{st}\left(\frac{a}{s^n}\right) + \mu_{st}\left(\frac{b}{s^m}\right) \,,$$

$$\text{and } \mu_{st}\left(\frac{a}{s^n}\right)\mu_{st}\left(\frac{b}{s^m}\right) = \frac{au^n}{t^n}\frac{bu^m}{t^m} = ab\frac{u^{n+m}}{t^{n+m}} = \mu_{st}\left(\frac{ab}{s^{n+m}}\right) \,,$$

$$\text{and } \mu_{st}(1_{A_s}) = \mu_{st}\left(\frac{1}{s^0}\right) = \frac{1u^0}{t^0} = 1_{A_t} \,. \tag{1.15.2}$$

If we then suppose that $t \leq w$ in S where $w = vt = vus$ for some $v \in S$, then we see that

$$\left(\mu_{tw} \circ \mu_{st}\right)\left(\frac{a}{s^n}\right) = \mu_{tw}\left(\frac{au^n}{t^n}\right) = \frac{av^nu^n}{w^n} = \mu_{sw}\left(\frac{a}{s^n}\right) \,, \tag{1.15.3}$$

as required. Importantly, we see that what we have done here is find a concrete example of 15.12 Example 15 from the lecture notes. That is, we have a poset $(S, \leq)$ and a category $\mathcal{C} = \mathsf{Rings}$, and we have specified a functor $F : \underline{(S, \leq)} \to \mathcal{C}$ which is equivalent to an object A_s in Rings and a morphism $\mu_{st} : A_s \to A_t$ whenever $s \leq t$ in S such that $\mu_{tw}\mu_{st} = \mu_{sw}$ whenever $s \leq t \leq w$.

Part c)

We finally want to show that the natural A-algebra homomorphism $\mathrm{colim}_{s\in S} A_s \to S^{-1}A$ is an isomorphism. We can do this by appealing to their respective universal properties. Recall the universal property of $S^{-1}A$ says that for a ring homomorphism $\ell_S : A \to S^{-1}A$ such that $\ell_S(S) \subseteq (S^{-1}A)^\times$, and if there is another ring homomorphism $\psi : A \to B$ such that $\psi(S) \subseteq B^\times$, then there is a uniquely induced ring homomorphism $S^{-1}\psi : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \ell_S}\downarrow & \nearrow_{\exists!} & \\ S^{-1}A & & \end{array} \,. \tag{1.15.4}$$

But this is precisely the universal property of the colimit! Identifying $F(s)$ with A_s for each $s \in S$ and an object $c = A_w$, we have a cocone diagram

$$\begin{array}{ccccc} & & A_w & & \\ & {\scriptstyle \mu_{su}}\nearrow & & \nwarrow{\scriptstyle \mu_{tw}} & \\ A_s & & \xrightarrow[\mu_{st}]{} & & A_t \end{array} \tag{1.15.5}$$

Then recall the universal property of the colimit: given a cone $(c, \{\psi_i\})$ for another cone $(c', \{\psi'_i\})$, there is a unique morphism $\mathrm{colim}_{s\in\underline{S}} A_s \to c'$ such that

$$\begin{array}{ccc} c & \longrightarrow & c' \\ \downarrow & \nearrow_{\exists!} & \\ \mathrm{colim}_{s\in\underline{S}} A_s & & \end{array} \,. \tag{1.15.6}$$

But under our correspondence above, that is precisely the universal property of $S^{-1}A$! Hence, since they satisfy the same universal property, we see that $\text{colim}_{s\in \underline{S}} A_s \simeq S^{-1}A$ as required. □

With reference to [1] and [7].

2 Projective Space

Q4. Local ring is dominated by a valuation ring

Let K be a field. If we have local subrings $(A, \mathfrak{m}_A), (B, \mathfrak{m}_B) \subseteq K$, we say that B dominates A if $A \subseteq B$ and $\mathfrak{m}_A = \mathfrak{m}_B \cap A$. We will show that if $(A, \mathfrak{m})$ is a local subring of K then it is dominated by a valuation ring V of K. Recall, V is a valuation ring if given $x \in K$ either x or $1/x$ is in V.

Let $\mathcal{C} = \{V_i\}_{i\in I}$ be a chain of local rings dominating A. We can then take $V = \bigcup_{i\in I} V_i$, of which we will find a maximal element to prove the claim. First note that it is clear that V is a subring of K since $\mathcal{C}$ is a chain, so closure will always hold. To show V is a local ring, recall that being V being a local ring is equivalent to having a maximal ideal $\mathfrak{m}_V$ such that every element of $V\backslash\mathfrak{m}_V$ is a unit, i.e. all of the nonunits of V are contained in $\mathfrak{m}_V$.

Suppose $x, y \in V$ are nonunits, then we will show that $x + y \in V$ is a nonunit. Since each chain element is a local ring, we must have that $x \in \mathfrak{m}_A$ and $y \in \mathfrak{m}_B$ for some $A, B \in \mathcal{C}$. Without loss of generality, we may assume via our domination property of V that $A \subseteq B$, hence we must have $x \in \mathfrak{m}_B$ and so $x+y \in B$. If $x+y \in V$ was a unit, then we would have some $C \in \mathcal{C}$ for which $(x + y)^{-1} \in C \subseteq V$, where $B \subseteq C$. But our domination property tells us that $\mathfrak{m}_B \subseteq \mathfrak{m}_C$, so $V\backslash\mathfrak{m}_B \supseteq V\backslash\mathfrak{m}_C$ so if $(x + y)^{-1}$ was in C then it would also be in B. But this contradicts the fact that $x+y \in \mathfrak{m}_B$, therefore $x+y \in V$ must be a nonunit.

Finally, we see that the set of nonunits in V forms an ideal: if $x \in V$ is a nonunit and $a \in V$, then xa is also a nonunit because if it wasn't then we would have $xa = u$ for some unit $u \in V$, but then $(u^{-1}a)x = 1$ contradicting the fact that x is a nonunit. Therefore, the set of nonunits in V form an ideal, meaning V is a local ring with maximal ideal $\mathfrak{m}_V$!

We then claim that V dominates every $A \in \mathcal{C}$. By definition we have $A \subseteq V$, so we just need to show that $\mathfrak{m}_A \subseteq \mathfrak{m}_V$. Suppose we have an $x \in \mathfrak{m}_a$ but $x \notin \mathfrak{m}_V$, that is, x is a nonunit in A but a unit in V. Then there is some $B \in \mathcal{C}$ such that x is a unit in B, so $x^{-1} \in B$. If A dominates B then we clearly have $x^{-1} \in A$ which contradicts $x \in \mathfrak{m}_A$, so suppose B dominates A, which means $\mathfrak{m}_A \subset \mathfrak{m}_B$. But since x is a nonunit in A, it also must be a nonunit in B, which is a contradiction! Therefore we must have that $x \in \mathfrak{m}_V$, thus $\mathfrak{m}_A \subseteq \mathfrak{m}_V$. In other words, we have shown that our chain $\mathcal{C}$ has an upper bound when considering the partial order on sets.

We can then appeal to Zorn's lemma, which says that the set of local rings containing A, i.e. our chain $\mathcal{C}$, has a maximal element V_M. We then appeal to a theorem (which I won't prove here) that states that if V_M is maximal on the ordering of local rings, then V_M is a valuation ring. In other words, we have found a valuation ring that dominates A!

With reference to [5] and [9]. (Note: apologies for the confusing and messy argument at the end, my brain is dead at the end of this very long and difficult semester).

Q5. Specialisation of spectra

Let A be a ring. Let $\mathfrak{p} \subseteq \mathfrak{q} \subseteq A$ be an inclusion of prime ideals of A. We will show that there is a valuation ring V and a ring homomorphism $s : A \to V$ such that $s^{-1}(0) = \mathfrak{p}$ and $s^{-1}(\mathfrak{m}) = \mathfrak{q}$, where $\mathfrak{m} \subseteq V$ is the unique maximal ideal. That is, if $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ (called specialisation), then it can be witnessed by a map Spec $V \to$ Spec A, where the closed point of Spec V maps to $\mathfrak{q}$ and its generic point goes to $\mathfrak{p}$.

We can appeal to Q4 to make sense of this. Consider the quotient field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \simeq \text{Frac}(A/\mathfrak{p})$ as our field K in Q4. Note that since $\mathfrak{p} \subseteq \mathfrak{q}$, we have $A_\mathfrak{q} \subseteq A_\mathfrak{p}$. Hence we can construct a ring homomorphism $s : A \to K$ as a composition of many other ring homomorphisms as follows:

$$s : A \xrightarrow{\phi} A_\mathfrak{q} \xrightarrow{t} A_\mathfrak{p} \xrightarrow{\pi} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \kappa(\mathfrak{p})\,. \tag{2.5.1}$$

We can consider the local ring $(A_\mathfrak{q}, \mathfrak{q}A_\mathfrak{q})$, where we note that this is a local ring since we know that the residue field is a field, hence $\mathfrak{q}A_\mathfrak{q}$ is the maximal ideal of $A_\mathfrak{q}$. If we then view the ring homomorphism $\pi \circ t : A_\mathfrak{q} \to \kappa(\mathfrak{p})$, we know that ring homomorphisms send local rings to local rings by the correspondence theorem. In other words, we know that the image $\pi(t(A_\mathfrak{q}))$ is a subring of $\kappa(\mathfrak{p})$ with maximal ideal $\pi(t(\mathfrak{q}A_\mathfrak{q}))$. By Q4, we know that there exists a valuation ring $(V, \mathfrak{m})$ that dominates the local ring $(\pi(t(A_\mathfrak{q})), \pi(t(\mathfrak{q}A_\mathfrak{q})))$, that is, $\pi(t(\mathfrak{q}A_\mathfrak{q})) = \mathfrak{m} \cap \pi(t(A_\mathfrak{q}))$. But then $\iota^{-1}(\pi(t(A_\mathfrak{q}))) = A_\mathfrak{q}$ in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, so $\iota^{-1}(\mathfrak{m}) = \pi(t(\mathfrak{q}A_\mathfrak{q}))$ where ι is the inclusion from V into $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Hence, we can calculate

$$\begin{aligned} s^{-1}(0) &= (\phi^{-1} \circ t^{-1} \circ \pi^{-1})(0) = (\phi^{-1} \circ t^{-1})(\mathfrak{p}A_\mathfrak{p}) = \phi^{-1}(\mathfrak{p}A_\mathfrak{p}) = \mathfrak{p}A = \mathfrak{p} \\ \text{and} \quad s^{-1}(\mathfrak{m}) &= (\phi^{-1} \circ t^{-1} \circ \pi^{-1})(\mathfrak{m}) = (\phi^{-1} \circ t^{-1})(\mathfrak{q}A_\mathfrak{q}) = \mathfrak{q}\,. \end{aligned} \tag{2.5.2}$$

Thus we have constructed the correct ring homomorphism $s : A \to V$ and so we are done. □

3 Stalks

Q1. Separated presheaf and stalks

Let X be a topological space and let $U \subseteq X$ be open with open cover $\{U_i \subseteq U\}_{i \in I}$. Let F be a separated presheaf on X, meaning the following map is injective:

$$F(U) \longrightarrow \prod_{i \in I} F(U_i)\,. \tag{3.1.1}$$

That is, for some other topological space Z, if $f, g : U \to Z$ are continuous and $f|_{U_i} = g|_{U_i}$ for all $i \in I$ then $f = g$. Define the stalk at $u \in U$ of F as

$$\begin{gathered} F_u = \{(f, U) : u \in U, f \in F(U)\}/\sim\,, \\ \text{where} \quad (f, U) \sim (g, V) \iff \exists \text{ open } x \in W \subseteq U \cap V \text{ with } f|_W = g|_W\,. \end{gathered} \tag{3.1.2}$$

We want to show that the following map is injective:

$$\begin{aligned} F(U) &\hookrightarrow \prod_{u \in U} F_u \\ f &\mapsto \prod_{u \in U} (f, U)\,. \end{aligned} \tag{3.1.3}$$

Suppose that $\prod_{u\in U}(f,U) = \prod_{u\in U}(g,U)$, so for each $u \in U$ we have $(f,U) = (g,U)$. Our equivalence relation then tells us that for each of these u we can find an open $W_u \subseteq U \cap U = U$ such that $u \in W_u$ and $f|_{W_u} = g|_{W_u}$. But since every u is contained in a W_u, that means we have found an open cover $\bigcup_{u\in U} W_u = U$. By our separation condition on F, the agreement of each $f|_{W_u} = g|_{W_u}$ gives us that $f = g$ and so we are done. □

With reference to [11].

Q5. Open subset of a scheme is also a scheme

Let X be a scheme, that is, a locally ringed space $(X, \mathcal{O}_X)$ such that for every point $x \in X$ there is an open subset $V \subseteq X$ with $x \in V$ such that the locally ringed space $(V, \mathcal{O}_X|_V)$ is isomorphic to an affine scheme $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$. Let $U \subseteq X$ be an open subset. We will show that $(U, \mathcal{O}_X|_U)$ is also a scheme.

Let $x \in U$. We want to show that there is an open neighbourhood $W \ni x$ that is isomorphic to an affine scheme. Since $X \ni x$ is a scheme, we can find a neighbourhood $V \subseteq X$ of x that is isomorphic to an affine scheme $\mathrm{Spec}A$ for some ring A. In defining this isomorphism as $\phi : V \to \mathrm{Spec}A$, we get that ϕ is also an isomorphism of topological spaces V and $\mathrm{Spec}A$. Clearly we will be looking to investigate the intersection of V and U to make our case.

Since $V \cap U$ is open and ϕ is an isomorphism (homeomorphism), we have that $\phi(V \cap U)$ is open in V. We know that the distinguished affine opens $\{D_A(a) | a \in A\}$ form a basis for the Zariski topology on $\mathrm{Spec}A$, so by the definition of a basis we can find some $a \in A$ such that $D_A(a) \subset \phi(V \cap U)$ and $x \in D_A(a)$.

From the Q3 Zariski Worksheet, we know that $D_A(a) \cong \mathrm{SpecA_a}$, hence meaning that $(D_A(a), \mathcal{O}_{\mathrm{Spec}A}\big|_{D_A(a)})$ is isomorphic to an affine scheme $\mathrm{SpecA_a}$. We can then use the inverse image functor ϕ^* from sheaves of commutative rings on $D_A(a)$ to sheaves of commutative rings on $\phi^{-1}(D_A(a))$ to show that $(\phi^{-1}(D_A(a)), \phi^*(\mathcal{O}_{\mathrm{Spec}A}\big|_{D_A(a)}))$ is also isomorphic to an affine scheme. By the construction of the inverse image functor, we have that $\phi^*(\mathcal{O}_{\mathrm{Spec}A}\big|_{D_A(a)})$ is isomorphic to $\mathcal{O}_U|_{\phi^{-1}(D_A(a))}$. But then

$$\phi^{-1}(D_A(a)) \subset \phi^{-1}(\phi(V \cap U)) = V \cap U \subset U\,. \tag{3.5.1}$$

That is, we can take $W = \phi^{-1}(D_A(a))$ and by the construction of the previous paragraph, we have that $(W, \mathcal{O}_X|_W)$ is isomorphic to an affine scheme. Therefore, $(U, \mathcal{O}_X|_U)$ is also a scheme!

(Note: I could see our lecture notes implicitly referred to this inverse image functor throughout, but never quite as explicit as the way in which I used it here. I didn't think it was necessary to prove big facts about ϕ^*, so I am merely appealing to the content I have read elsewhere.)

With reference to [8].

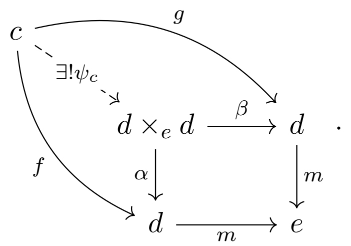
4 Limits

Q3. Monomorphisms and epimorphisms

Let $\mathcal{C}$ be a category. A morphism $m : d \to e$ is a monomorphism if for two morphisms $f : c \to d$ and $g : c \to d$, if the compositions to e agree, so $m \circ f = m \circ g$, then the morphisms agree, so $f = g$.

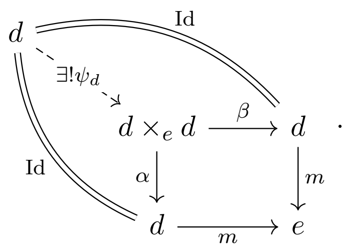
Part a)

We will show that $m : d \to e$ is a monomorphism if and only if $\psi_d : d \to d \times_e d$ is an isomorphism. Recall that this fiber product of $m : d \to e$ with itself is defined by the universal property, meaning $d \times_e d$ comes equipped with two morphisms $\alpha, \beta : d \times_e d \to d$ such that for any other triple (c, f, g) where $f, g : c \to d$ are morphisms with $m \circ f = m \circ g$, there exists a unique $\psi_c : c \to d \times_e d$. That is, the following diagram commutes:



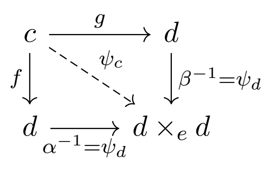
(4.3.1)

In particular, we can take $c \equiv d$ in our triplet, and $f, g \equiv \mathrm{Id}$, so the universal property gives us $\psi_d : d \to d \times_e d$, which is unique. That is, we now have another commutative diagram:



(4.3.2)

Suppose that $\psi_d : d \to d \times_e d$ as in (4.3.2) is an isomorphism, so $\psi_d^{-1} : d \times_e d \to d$ exists. Since we have $\alpha \circ \psi_d = \mathrm{Id}$, applying both sides to ψ_d^{-1} we see that we must have $\alpha = \psi_d^{-1}$, and similarly we must have $\beta = \psi_d^{-1}$. Then suppose that (4.3.1) commutes for some triple (c, f, g), so $m \circ f = m \circ g$. Then using the established inverse facts, we have a diagram



(4.3.3)

This tells us that $\psi_d \circ f = \psi_d \circ g$, and since ψ_d is an isomorphism, we can apply the inverse to both sides to see that $f = g$, hence m is a monomorphism.

Suppose now that $m : d \to e$ is a monomorphism and that we are in the situation of (4.3.2). Then we have

$$m \circ \alpha \circ \psi_d = m \circ \mathrm{Id}_d\,, \qquad \text{so} \quad \alpha \circ \psi_d = \mathrm{Id}_d\,. \tag{4.3.4}$$

Then following the bottom left triangle of (4.3.2), we have a map $\psi_d \circ \alpha : d \times_e d \to d \times_e d$. Applying the universal property (4.3.1) to $c \equiv d \times_e d$, this map is unique. In particular, we must have

$$\alpha \circ (\psi_d \circ \alpha) = (\alpha \circ \psi_d) \circ \alpha = \alpha\,, \qquad \text{meaning} \quad \psi_d \circ \alpha = \mathrm{Id}_{d \times_e d}\,. \tag{4.3.5}$$

Since α is a left and right inverse to ψ_d, we clearly must have a well defined $\psi_d^{-1} = \alpha$, hence meaning that ψ_d is an isomorphism as required. □

Part b)

A morphism $d \to e$ is an epimorphism if it is a monomorphism in $\mathcal{C}^{\text{opp}}$. We will show that $d \to e$ is an epimorphism if and only if $e \coprod_d e \to e$ is an isomorphism. But thankfully the fibered coproduct $e \coprod_d e$ is precisely the categorically dual notion of the fiber product, that is, it satisfies the same universal property in $\mathcal{C}^{\text{opp}}$, meaning all arrows are reversed. Therefore we see that our proof in part a) tells us precisely that $e \coprod_d e \to e$ is an epimorphism!

With reference to [4].

Q6. Morphisms of restricted presheaves

Let X be a topological space. Let $\alpha : F \to G$ be a morphism (natural transformation) of presheaves on X. For each $U \subseteq X$ open, let $\alpha_U : F(U) \to G(U)$ be the induced morphism.

Part a)

We will first show that α is a monomorphism of presheaves if and only if α_U is injective for all $U \subseteq X$ open. Suppose α_U is injective. Let $\psi, \pi : H \to F$ be morphisms of presheaves such that $\alpha\psi = \alpha\pi$, then we want to show that $\psi = \pi$. In particular, for $U \subset X$ open and $x \in H(U)$ we can simply rewrite this as $(\alpha\psi)(U)(x) = (\alpha\pi)(U)(x)$, and since α is a natural transformation, this gives us $\alpha(U)(\psi(U)(x)) = \alpha(U)(\pi(U)(x))$. Since $\alpha_U = \alpha(U)$ is injective, we must therefore have $\psi(U)(x) = \pi(U)(x)$. Since this is true for all objects U, we have that $\psi = \pi$ as morphisms.

Now assume that $\alpha : F \to G$ is a monomorphism, we want to show that $\alpha_U : F(U) \to G(U)$ is injective. Suppose that $\alpha_U(x) = \alpha_U(y)$ for some $x, y \in F(U)$. Taking our cues from Categories Q8 and lecture notes, we can define a presheaf H for some $V \subseteq X$ as

$$H(V) = \begin{cases} \{*\} & \text{if there is an arrow } \gamma : V \to U \text{ in the category } X \\ \varnothing & \text{otherwise} \end{cases}\,. \tag{4.6.1}$$

Now suppose we have morphisms $\psi, \pi : H \to F$. Note then that by definition we have that $\psi(U)$ and $\pi(U)$ are only restricted to act on $\{*\}$ since there is always an identity arrow $U \to U$, hence we can define our morphisms to act as $\psi(U)(*) = x$ and $\pi(U)(*) = y$. Then since we have $\alpha_U(x) = \alpha_U(y)$, this becomes $\alpha_U(\psi(U)(*)) = \alpha_U(\psi(U)(*))$, hence $(\alpha\psi)(U)(*) = (\alpha\pi)(U)(*)$. But since α is a monomorphism, this implies that $\psi(U)(*) = \pi(U)(*)$ and so since they agree on all U, we must have $\psi = \pi$ as required. □

Part b)

We then want to show that α is an epimorphism of presheaves if and only if α_U is surjective for all $U \subset X$ open. But, we have done all of the necessary work because surjectivity and epimorphisms are the categorical dual of injectivity and monomorphisms (in the category of Sets). That is, if one simply reverses all of the arrows in the previous argument, then the statement will hold in the opposite category, hence giving us the desired statement. Yay for duality!

With reference to [2].

5 Adjoints

Q1. Adjoints preserve limits and colimits

Let $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ be an adjoint pair (I have changed the notation from the sheet slightly). We will show that L preserves colimits and that R preserves limits. That is, given another small category $\mathcal{I}$ and functors $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{I} \to \mathcal{D}$ we have

$$\underset{\mathcal{I}}{\operatorname{colim}}(L \circ F) \xrightarrow{\sim} L\left(\underset{\mathcal{I}}{\operatorname{colim}}\, F\right) \quad \text{and} \quad R\left(\lim_{\mathcal{I}} G\right) \xrightarrow{\sim} \lim_{\mathcal{I}}(R \circ G)\,. \tag{5.1.1}$$

We start with the first one and first recall the definition (i.e. the universal property) of a colimit. A colimit of $F : \mathcal{I} \to \mathcal{C}$ is an object $\underset{\mathcal{I}}{\operatorname{colim}}\, F \in \mathcal{C}$, together with morphisms $s_i : F(i) \to \underset{\mathcal{I}}{\operatorname{colim}}\, F$ such that

- For a morphism $\phi : i \to j$ in $\mathcal{I}$ we have $s_i = s_j \circ F(\phi)$ and;
- For any object W in $\mathcal{C}$ and a family of morphisms $t_i : F(i) \to W$ indexed by $i \in \mathcal{I}$ such that for all $\phi : i \to j$ we have $t_i = t_j \circ F(\phi)$, there exists a unique morphism $t : \underset{\mathcal{I}}{\operatorname{colim}}\, F \to W$ such that $t_i = t \circ s_i$ for every object $i \in \mathcal{I}$. We can alternatively write this as:

$$\operatorname{Hom}_{\mathcal{C}}(\underset{\mathcal{I}}{\operatorname{colim}}\, F, W) \simeq \left\{ (t_i)_{i\in\mathcal{I}} \;\middle|\; \begin{array}{l} t_i : F(i) \to W\,, \\ \text{for all } \ \phi : i \to j\,, \text{we have } t_i = t_j \circ F(\phi) \end{array} \right\}. \tag{5.1.2}$$

We claim that $L\left(\underset{\mathcal{I}}{\operatorname{colim}}\, F\right)$ is a (and therefore by the universal property the unique) colimit of $L \circ F$. Let $s_i : F(i) \to \underset{\mathcal{I}}{\operatorname{colim}}\, F$ be the associated maps to F, then we have a family of morphisms $t_i : (L \circ F)(i) \to L(\underset{\mathcal{I}}{\operatorname{colim}}\, F)$, where $t_i = L \circ s_i$, which satisfy the necessary property since it is just the composition of maps. Hence, via the universal property, we must have a canonical morphism $t : \underset{\mathcal{I}}{\operatorname{colim}}\, L \circ F$ which corresponds to the family of morphisms t_i. We shall denote this family of morphisms as $L(F(i) \to \underset{\mathcal{I}}{\operatorname{colim}}\, F)$.

We then want to show that our construction satisfies the universal property. Let $d \in \mathcal{D}$ be an object. Then by definition of adjoint pair we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}}(L(\underset{\mathcal{I}}{\operatorname{colim}}\, F), d) &\simeq \operatorname{Hom}_{\mathcal{C}}(\underset{\mathcal{I}}{\operatorname{colim}}\, F, R(d)) \\ &\simeq \left\{ (t_i)_{i\in\mathcal{I}} \;\middle|\; \begin{array}{l} t_i : F(i) \to R(d)\,, \\ \text{for all } \ \phi : i \to j\,, \text{we have } t_i = t_j \circ F(\phi) \end{array} \right\} \\ &\simeq \left\{ (t'_i)_{i\in\mathcal{I}} \;\middle|\; \begin{array}{l} t'_i : L(F(i)) \to d\,, \\ \text{for all } \ \phi : i \to j\,, \text{we have } t'_i = t'_j \circ L \circ F(\phi) \end{array} \right\} \\ &\simeq \operatorname{Hom}_{\mathcal{D}}(\underset{\mathcal{I}}{\operatorname{colim}}\, L \circ F, d)\,. \end{aligned} \tag{5.1.3}$$

Hence under this isomorphism we have

$$\big(t' : L(\operatorname*{colim}_{\mathcal{I}} F) \to d\big) \longmapsto \Big(t' \circ L\big(F(i) \to \operatorname*{colim}_{\mathcal{I}} F\big)\Big)_{i\in\mathcal{I}} . \tag{5.1.4}$$

Thus, this shows that $L(\operatorname*{colim}_{\mathcal{I}} F)$ with $L(F(i) \to \operatorname*{colim}_{\mathcal{I}} F)$ satisfies the necessary universal property, and thus is a (the) colimit of $L \circ F$. □

Note that, as with previous questions, G preserving limits will be the exact same argument but with every arrow reversed, so we won't repeat it.

With reference to [6]. (Note, I am not taking Christian's class, hence why I didn't feel guilty in doing this question).

Q6. Adjoint fully faithful if and only if unit is isomorphism

Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjoint pair. Recall the unit and counit map

$$\begin{aligned} &\eta : \mathrm{id}_{\mathcal{C}} \implies GF\,, &&\text{and} && \varepsilon : FG \implies \mathrm{id}_{\mathcal{D}} \\ \text{which induces}\quad &\eta_c : c \to GF(c) &&\text{and} && \varepsilon_c : FG(c) \to c \end{aligned} \tag{5.6.1}$$

We will show that F is fully faithful if and only if the induced map η_c is an isomorphism for all objects c in $\mathcal{C}$. Recall that a functor F is fully faithful if for every $c, c' \in \mathcal{D}$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(F(c), F(c')) \tag{5.6.2}$$

is bijective.

Suppose F is fully faithful and suppose we have another morphism $c' \to GF(c)$ for $c, c' \in \mathcal{C}$. Then by the definition of adjunction, this corresponds to another morphism $F(c') \to F(c)$. But since F is fully faithful, as can be seen from the above definition, this corresponds to a morphism $c \to c'$. Hence we have a bijection (and therefore an isomorphism)

$$\mathrm{Hom}_{\mathcal{C}}(c', c) \simeq \mathrm{Hom}_{\mathcal{C}}(c', GF(c)) \tag{5.6.3}$$

which is defined by $\eta_c : c \to GF(c)$, hence η_c is an isomorphism.

Now suppose η_c is an isomorphism, where means the composition map

$$\mathrm{Hom}_{\mathcal{C}}(c', c) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(F(c'), F(c)) \xrightarrow{g} \mathrm{Hom}_{\mathcal{C}}(GF(c'), GF(c)) \tag{5.6.4}$$

is bijective between the end sets. Therefore for $x, y \in \mathrm{Hom}_{\mathcal{C}}(F(c'), F(c))$, if $f(x) = f(y)$ then we can just take $(g \circ f)^{-1}(f(x)) = x = y = (g \circ f)^{-1}(f(y))$ and so f is injective, showing that F is faithful.

Suppose we have a morphism $\gamma : F(c') \to F(c)$, we want to find its preimage under f to show that F is full. Recall from Q3 adjoints that η and ε induced natural transformations $F \implies FGF \implies F$ and $G \implies GFG \implies G$ which are both their respective identities. This tells us how to construct a preimage, namely we can take

$$\alpha = \eta_{c'}^{-1} \circ G(\gamma) \circ \eta_c\,. \tag{5.6.5}$$

Doing a quick calculation we see that

$$
\begin{aligned}
GF(\alpha)\circ\eta_c &= GF(\eta_{c'}^{-1}\circ G(\gamma)\circ\eta_c)\circ\eta_c \\
&= GF(\eta_{c'}^{-1})\circ GFG(\gamma)\circ GF(\eta_c)\circ\eta_c \\
&= \eta_{c'}\circ\gamma \\
&= G(\gamma)\circ\eta_c
\end{aligned} \tag{5.6.6}
$$

which shows that we must have $G(\alpha)=\gamma$! Therefore, F is fully faithful as required. □

With reference to [3] and [10].

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