Lie Algebras Assignment 3

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Background 1

Q1. Exponentials Acting

Let V be a normed space over a field $\mathbb{F}$ and $v \in V$, where we may also view $\mathbb{F}$ as a normed space with $\|\lambda\|_{\mathbb{F}} = |\lambda|$ for any $\lambda \in \mathbb{F}$. Define the (obviously) linear transformation

$$\eta_v : \mathbb{F} \longrightarrow V\,, \quad \eta_v(\lambda) = \lambda v\,. \tag{1.1}$$

Part a)

Recalling that the definition of a norm gives us $\|\eta_v(\lambda)\|_V = \|\lambda v\|_V = |\lambda|\|v\|_V$, by definition we have

$$\begin{aligned}\|\eta_v\|_{\mathcal{B}(\mathbb{F},V)} &= \sup\left\{\frac{\|\eta_v(\lambda)\|_V}{\|\lambda\|_{\mathbb{F}}} \;\middle|\; \lambda \in \mathbb{F} \text{ s.t. } \lambda \neq 0\right\} = \sup\left\{\frac{|\lambda|\|v\|_V}{|\lambda|} \;\middle|\; \lambda \in \mathbb{F} \text{ s.t. } \lambda \neq 0\right\} \\ &= \sup\left\{\|v\|_V \;\middle|\; \lambda \in \mathbb{F} \text{ s.t. } \lambda \neq 0\right\} = \|v\|_V\,,\end{aligned} \tag{1.2}$$

thus proving the identity.

Part b)

We will show that there is a norm preserving isomorphism of vector spaces

$$\psi : V \longrightarrow \mathcal{B}(\mathbb{F}, V)\,, \quad v \longmapsto \eta_v\,. \tag{1.3}$$

Part a) gave us this this map is indeed norm preserving since $\|v\| = \|\eta_v\|$. The map is clearly linear since for all $\lambda \in \mathbb{F}$ we have

$$\psi(\alpha v + \beta w)(\lambda) = \eta_{\alpha v + \beta w}(\lambda) = (\alpha v + \beta w)\lambda = \lambda\alpha v + \lambda\beta w = (\alpha\eta_v + \beta\eta_w)(\lambda)\,. \tag{1.4}$$

To show it is bijective, we claim that

$$\phi : \mathcal{B}(\mathbb{F}, V) \longrightarrow V\,, \quad f \longmapsto f(1) \tag{1.5}$$

is the unique inverse to ψ. We compute, for all $\lambda \in \mathbb{F}$,

$$\begin{aligned}&(\phi \circ \psi)(v) = \phi(\eta_v) = \eta_v(1) = v\,, \text{ so } \phi \circ \psi = 1_V\,, \\ \text{and} \quad &(\psi \circ \phi)(f)(\lambda) = \psi(f(1))(\lambda) = \eta_{f(1)}(\lambda) = \lambda f(1) = f(\lambda)\,, \text{ so } \psi \circ \phi = 1_{\mathcal{B}(\mathbb{F},V)}\end{aligned} \tag{1.6}$$

where the last equality follows from the fact that f is linear over $\mathbb{F}$, thus showing that ϕ is inverse to ψ and hence the unique inverse, thus showing that ψ is bijective and hence a norm-preserving isomorphism of vector spaces. Indeed, it is also a norm-preserving isomorphism of normed spaces (i.e. ψ and ϕ are both continuous), which follows from a simple calculation to find $\|\psi\| = 1$ and $\|\phi\| = 1$.

Part c)

Let V and W be normed spaces. Consider the evaluation map

$$\begin{aligned} \Phi : \mathcal{B}(V, W) \times V &\longrightarrow W \\ (T, v) &\longmapsto T(v) \,. \end{aligned}$$

In part b) we showed that $V \cong \mathcal{B}(\mathbb{F}, V)$ via a norm preserving continuous map ψ, which we may easily extend to a continuous map on the product acting as the identity on $\mathcal{B}(V, W)$ that makes the following diagram commute

$$\begin{array}{ccc} \mathcal{B}(V, W) \times V & \xrightarrow{\quad \Phi \quad} & W \\ {\scriptstyle \cong} \big\downarrow & & \big\uparrow {\scriptstyle \cong} \\ \mathcal{B}(V, W) \times \mathcal{B}(\mathbb{F}, V) & \xrightarrow{\ \Psi \ } & \mathcal{B}(\mathbb{F}, W) \end{array} \,. \tag{1.7}$$

We know from Lemma B1-8 that the map Ψ is continuous. Therefore we see that Φ is a composition of continuous maps and so is itself continuous.

Part d)

Let $T : V \to V$ be a bounded linear operator on a Banach space V. We know from Theorem B1-7 that

$$\exp(T) = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{1}{n!} T^n = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

converges absolutely in $\mathcal{B}(V, V)$, that is $\exp(T) \in \mathcal{B}(V, V)$. We know by Lemma B1-8 that for any fixed $m \in \mathbb{N}$ the function $\sum_{n=0}^{m} \frac{1}{n!} T^n$ is in $\mathcal{B}(V, V)$ as it is just a composition and sum of $T \in \mathcal{B}(V, V)$. Part c) gave us continuity of Φ, hence for a fixed $v \in V$ we have

$$\begin{aligned} \exp(T)(v) = \Phi(\exp(T), v) &= \Phi \left(\lim_{m \to \infty} \sum_{n=0}^{m} \frac{1}{n!} T^n, v \right) \\ &= \lim_{m \to \infty} \Phi \left(\sum_{n=0}^{m} \frac{1}{n!} T^n, v \right) = \lim_{m \to \infty} \left(\sum_{n=0}^{m} \frac{1}{n!} T^n(v) \right) \,, \end{aligned} \tag{1.8}$$

where we could pull the limit outside due to the continuity of Φ, thus showing the desired identity. □

Q2. Trace vs determinant

We will prove that for any $X \in M_n(\mathbb{C})$ the trace-determinant identity holds, that is

$$\exp(\mathrm{tr}(X)) = \det\exp(X)\,. \tag{2.1}$$

Since our matrix is over the algebraically closed $\mathbb{C}$, by the Jordan normal form theorem, we may write $X = P^{-1}JP$ for some invertible change of basis matrix P and a Jordan normal form matrix J, where

$$J = \begin{pmatrix} J_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & J_k \end{pmatrix}, \quad \text{where} \quad J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \ddots & \lambda_i \end{pmatrix} \tag{2.2}$$

for eigenvalues λ_i of X for $1 \leq i \leq k$ (possibly non-distinct), where the dimensions of the Jordan block and the number of Jordan blocks are determined by quantities like the algebraic multiplicity of the eigenvalues. By Exercise B1-3, we know that since P is a norm-preserving isomorphism of vector spaces (since it is just a change of basis matrix) we have

$$\begin{aligned} \exp(X) = \exp(P^{-1}JP) = P^{-1}\exp(J)P\,, \\ \text{so} \quad \det\exp X = \det(P^{-1}\exp(J)P) = \det\exp J\,, \end{aligned} \tag{2.3}$$

since $\det P^{-1} = (\det P)^{-1}$ and det is a homomorphism. Therefore we may restrict our attention to calculating $\exp(J)$. Since J is block diagonal we have

$$J^n = \begin{pmatrix} J_1^n & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & J_k^n \end{pmatrix} \tag{2.4}$$

We know from lectures that we can decompose $J_i = \lambda_i I + N$ for some nilpotent N where we set $k = \inf\{k \in \mathbb{N} : N^k = 0\}$. Then we have

$$\exp(J_i) = e^{\lambda_i} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \dots & \frac{1}{(k-1)!} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \frac{1}{2} \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =: e^{\lambda_i} T_i\,. \tag{2.5}$$

We then see that

$$\begin{aligned} \exp(J) &= \sum_{n=0}^{\infty} \frac{1}{n!} J^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} J_1^n & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & J_k^n \end{pmatrix} \\ &= \begin{pmatrix} \exp J_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \exp J_k^n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} T_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & e^{\lambda_k} T_k \end{pmatrix} \end{aligned} \tag{2.6}$$

is an upper triangular matrix due to our calculation in (2.5). But by performing a simple cofactor expansion we know that the determinant of an upper triangular matrix is just the product of the diagonal entries. For notational ease we may rewrite the non-distinct eigenvalues as λ_j for $1 \leq j \leq n$ (instead of trying to account for the varying algebraic multiplicities of the Jordan blocks and such), and so we have

$$\det \exp X = \det \exp J = e^{\lambda_1} \dots e^{\lambda_n} = e^{\sum_{i=1}^{n} \lambda_i} = e^{\mathrm{tr} J} = e^{\mathrm{tr}(P^{-1}JP)} = e^{\mathrm{tr} X} \,, \tag{2.7}$$

where the second last equality follows the cyclicity of the trace, i.e. $\mathrm{tr}(P^{-1}JP) = \mathrm{tr}(PP^{-1}J) = \mathrm{tr}J$, which thus proves the desired identity. □

Q3. Heisenberg

Define the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.1}$$

and let $\alpha \in \mathbb{R}$. We see that we have

$$X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad H^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.2}$$

hence each of X, Y and H are nilpotent of degree 2. Therefore we have

$$\exp(\alpha X) = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha X)^n = I_3 + \alpha X + \sum_{j=0}^{\infty} \frac{1}{(j+2)!} \alpha^{2+j} X^{2+j} = I_3 + \alpha X = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.3}$$

and so we similarly have

$$\exp(\alpha Y) = I_3 + \alpha Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} , \quad \text{and} \quad \exp(\alpha H) = I_3 + \alpha H = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.4}$$

We note that $\exp(\alpha X)$ and friends are shear matrices, which suggests that $X, Y, H \in M_n(\mathbb{C})$ generate shear matrices in $\mathrm{GL}_n(\mathbb{C})$ via the exponential map. □

Lecture 5

Q4. Skew vs unitary

Let $\mathcal{H}$ be a finite dimensional inner product space and $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. We want to show that T is skew self-adjoint if and only if $e^{\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$. Lemma L5-4 tells us that T is self adjoint if and only if $e^{i\alpha T}$ is unitary for every $\alpha \in \mathbb{R}$, which can be translated into: $(-iT)$ is self adjoint if and only if $e^{i\alpha(-iT)} = e^{\alpha T}$ is unitary. But if $(-iT)$ is self adjoint then (noting that we have linearity in the second entry and conjugate linearity in the first entry of $\langle , \rangle$) we have

$$\langle (-iT)x, y \rangle = \langle x, (-iT)y \rangle \,, \quad \text{so} \quad \overline{(-i)} \langle Tx, y \rangle = (-i) \langle x, Ty \rangle \,,$$
$$\text{so} \quad \langle Tx, y \rangle = -\langle x, Ty \rangle \,, \tag{4.1}$$

thus showing that $(-iT)$ is self adjoint if and only if T is skew self-adjoint. Therefore T is skew self-adjoint if and only if $e^{\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$. □