Advanced Methods: Transforms Assignment 2

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Q1. Fourier and Laplace transforms and convolutions

Part a)

We first derive an analogue of the derivative property of a Fourier transform for Fourier-sine and Fourier-cosine transforms. For a function $f(x)$ with all necessary properties to take Fourier tranforms, i.e. substantial decay at $\pm\infty$, integrability, etc., integration by parts gives us

$$\begin{aligned}\mathcal{F}_c\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f'(x)\cos(kx)dx \\ &= \frac{1}{\sqrt{2\pi}}\left[[f(x)\cos(kx)]_0^\infty + k\int_0^\infty f(x)\sin(kx)dx\right] \\ &= \frac{f(0)}{\sqrt{2\pi}} + k\mathcal{F}_s\{f(x)\}\,, \end{aligned} \tag{1.1}$$

and similarly

$$\begin{aligned}\mathcal{F}_s\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f'(x)\sin(kx)dx \\ &= \frac{1}{\sqrt{2\pi}}\left[[f(x)\sin(kx)]_0^\infty - k\int_0^\infty f(x)\cos(kx)dx\right] \\ &= -k\mathcal{F}_c\{f(x)\}\,. \end{aligned} \tag{1.2}$$

Part b)

To establish the properties of $f(x) : [0,\infty) \to \mathbb{R}$ in order to satisfy the given equations, we calculate using our derived properties in part a),

$$\mathcal{F}_c\{f''(x)\} = \frac{f'(0)}{\sqrt{2\pi}} + k\mathcal{F}_s\{f'(x)\} = \frac{f'(0)}{\sqrt{2\pi}} - k^2\mathcal{F}_c\{f(x)\}\,, \tag{1.3}$$

$$\text{and} \quad \mathcal{F}_s\{f''(x)\} = -k\mathcal{F}_c\{f'(x)\} = \frac{-kf(0)}{\sqrt{2\pi}} - k^2\mathcal{F}_s\{f(x)\}\,. \tag{1.4}$$

We first notice that in order to get the desired equalities we clearly need $f'(0) = f(0) = 0$. Further to this, we see throughout the process that we are assuming the existence of Fourier transforms for $f(x)$, $f'(x)$ and $f''(x)$. Therefore to satisfy the requirements of Fourier's Integral Theorem (which are sufficient but not necessary conditions), we must have that $f(x)$ is piecewise C^3 and that each of $f(x)$, $f'(x)$ and $f''(x)$ are absolutely integrable.

Part c)

Let $f(t) = t^\alpha$ and $g(t) = t^\beta$ for real $\alpha, \beta > -1$. We will calculate the Laplace convolution $f * g$ where

$$(f * g)(x) = \int_0^x f(t)g(x-t)dt = \int_0^x t^\alpha (x-t)^\beta dt\,. \tag{1.5}$$

Using formulas provided on the formula sheet we have

$$\mathcal{L}\{(f*g)(x)\} = \mathcal{L}\{f\}\mathcal{L}\{g\} = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\frac{\Gamma(\beta+1)}{p^{\beta+1}} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{p^{(\alpha+\beta+1)+1}}\,. \tag{1.6}$$

Then, using the fact that for $a \in \mathbb{R}\backslash\mathbb{Z}_-$ we have $\mathcal{L}^{-1}\{\frac{1}{p^{a+1}}\} = \frac{1}{\Gamma(a+1)}x^a$, we can hence calculate

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{p^{(\alpha+\beta+1)+1}}\right\} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma((\alpha+1)+(\beta+1))}x^{\alpha+\beta+1} = B(\alpha+1,\beta+1)x^{\alpha+\beta+1}\,. \tag{1.7}$$

Part d)

Let $T > 0$ be a constant and $f : [0,\infty)$ be a function that is T-periodic, so

$$f(t+T) = f(t), \qquad \text{hence} \quad f(t) = f(t-T) \quad \text{for all } t \geq 0\,. \tag{1.8}$$

We can then define a new function $f_T(t) = f(t)$ for $0 \leq t \leq T$ and $f_T(t) = 0$ for $t > T$. We can rewrite this definition of $f_T : [0,\infty)$ using Heaviside step functions as

$$f_T(t) = [1 - H(t-T)]f(t) = f(t) - H(t-T)f(t-T)\,. \tag{1.9}$$

Then we can calculate, using the linearity of Laplace transforms and the formula sheet again,

$$\mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t) - H(t-T)f(t-T)\} = \mathcal{L}\{f(t)\} - e^{-pT}\mathcal{L}\{f(t)\}\,, \tag{1.10}$$

hence arriving at our desired formula,

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_T(t)\}}{1-e^{-pT}}\,. \tag{1.11}$$

Q2. A falling rope

Let $c > 0$ be a constant and g the acceleration due to gravity. A rope lies at rest along the x-axis, stretching from 0 to infinity. At time $t = 0$, the support is removed and gravity pulls the rope down. During the whole time the left end of the rope is fixed at $(x, h) = (0, 0)$. We may assume that the displacement $h(x, t)$ of the string is described by the wave equation

$$\ddot{h}(x,t) = c^2 h''(x,t) - g \qquad (\text{with } x, t > 0)\,. \tag{2.1}$$

Part a)

The initial and boundary conditions of the desired setup can be specified as

$$h(x,0) = 0\,, \qquad \frac{\partial h}{\partial t}(x,0) = 0\,, \tag{2.2}$$

$$h(0,t) = 0\,, \qquad \lim_{x\to\infty} \frac{\partial h}{\partial x}(x,t) = 0\,. \tag{2.3}$$

The first two are given by the fact that the rope is still on the x-axis at $t = 0$. The second two are given by the fact that the rope is fixed at $x = 0$, and importantly that the shape of the rope falling at infinity will still be flat as we expect the shape at any time to have a concave up decreasing shape.

Part b)

We note that since we are working in a positive time and space domain $t, x > 0$ and the initial values behave well at the origin that we are best off using Laplace transforms in the time domain, reducing the problem to a spatial differential equation. We first define the Laplace transform of $h(x, t)$ with respect to t,

$$H(x,p) = \int_0^\infty h(x,t) e^{-pt} dt\,. \tag{2.4}$$

We can then calculate

$$\begin{aligned} c^2 \frac{\partial^2 H}{\partial x^2} &= c^2 \frac{\partial^2}{\partial x^2} \int_0^\infty h(x,t) e^{-pt} dt \\ &= \int_0^\infty c^2 \frac{\partial^2 h}{\partial x^2} e^{-pt} dt \\ &= \int_0^\infty \left(\frac{\partial^2 h}{\partial t^2} + g \right) e^{-pt} dt \\ &= p^2 H(x,p) - p h(x,0) - h_t(x,0) + \frac{g}{p} \\ &= p^2 H(x,p) + \frac{g}{p}\,. \end{aligned} \tag{2.5}$$

In the fourth line we used the fact that $\mathcal{L}\{f''(x)\} = p^2 L(p) - p f(0) - f'(0)$ and in the fifth line we were able to impose our initial conditions. This then gives us a standard linear second order ordinary differential equation in x,

$$\frac{\partial^2 H}{\partial x^2} - \frac{p^2}{c^2} H = \frac{g}{c^2 p} \,. \tag{2.6}$$

This then yields the simple solution, where A and B are constants,

$$H(x,p) = A \exp\left(\frac{p}{c}x\right) + B \exp\left(-\frac{p}{c}x\right) - \frac{g}{p^3} \,. \tag{2.7}$$

Imposing our two boundary conditions gives us

$$H(0,p) = A + B - \frac{g}{p^3} = 0 \,, \tag{2.8}$$

$$\text{and} \quad \lim_{x\to\infty} H_x(x,p) = \lim_{x\to\infty} \left\{ \frac{Ap}{c} \exp\left(\frac{p}{c}x\right) - \frac{Bp}{c} \exp\left(-\frac{p}{c}x\right) \right\} = 0 \,, \tag{2.9}$$

so we have $A = 0$ and $B = g/p^3$. Hence, our solution for H is

$$H(x,p) = \exp\left(-\frac{1}{c}xp\right) \frac{g}{p^3} - \frac{g}{p^3} \,. \tag{2.10}$$

We can then take the inverse Laplace transform of this, in particular noting the properties $\mathcal{L}\{\Theta(t-a)g(t-a)\} = \exp(-ap)\mathcal{L}\{g(t)\}$ (where Θ is the Heaviside step function so as to not confuse notation), and $\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{p^{a+1}}$, which leads to

$$h(x,t) = \mathcal{L}^{-1}\{H(x,p)\} = \frac{g}{\Gamma(3)} \Theta\left(t - \frac{1}{c}x\right) \left(t - \frac{1}{c}x\right)^2 - \frac{g}{\Gamma(3)} t^2 \,. \tag{2.11}$$

With suitable rearrangement, we can write our final solution as

$$h(x,t) = \begin{cases} \frac{g}{2c^2} x(x - 2ct) & \text{if } x < ct \\ -\frac{g}{2} t^2 & \text{if } x \geq ct \end{cases} \,. \tag{2.12}$$

Part c)

We can see from our solution in (2.12) that for a fixed $t = \tau$, we have a positive quadratic from $0 < x < c\tau$, which agrees with our concave up decreasing shape, and for $x > c\tau$ the rope remains flat parallel to the x-axis. A plot of the situation for this fixed $t = \tau$ is below.

Note the trajectory of the rope from $\tau - 1 < t < \tau + 1$, where particles closer to the origin get closer and closer to the "wall" at $x = 0$ as time progresses. We see that the "wave" propagates through the rope as more particles start to wrap back towards the wall as time progresses.

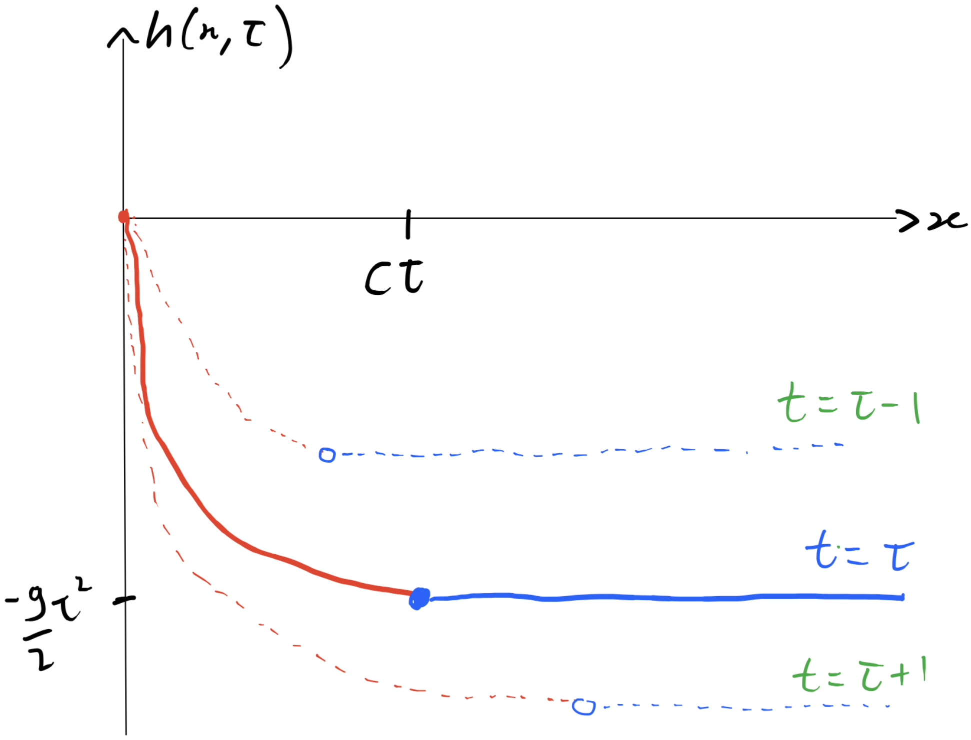

Figure 2.1: A plot of the shape of the rope for a fixed $t = \tau > 0$

Q3. Closer inspection of an integral equation

In the lectures we have analysed the integral equation

$$f(x) = e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy \tag{3.1}$$

which, after taking Fourier transforms, we found could be formally expressed as

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2 - (2\lambda - 1)} \,. \tag{3.2}$$

Part a)

We will analyse the case $\lambda > 1/2$, hence inducing singularities on the real axis. We thus consider the contour integral

$$I = \oint \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz \,. \tag{3.3}$$

In the case of $x \geq 0$, we perform this integral in the anti-clockwise direction around the semi circular contour in the *upper-half plane* (this is important), with indented clock-wise semi-circles around the poles at $z_\pm = \pm\sqrt{2\lambda - 1}$ on the real axis. Cauchy's theorem gives us

$$\begin{aligned} I = &\left(\int_{-R}^{z_- - r_1} + \int_{z_- + r_1}^{z_+ - r_2} + \int_{z_+ + r_2}^{R} \right) \frac{e^{ixk}}{k^2 - (2\lambda - 1)} dk + \lim_{r_1 \to 0} \int_{C_{r_1}} \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz \\ &+ \lim_{r_2 \to 0} \int_{C_{r_2}} \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz + \int_{C_R} \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz = 0 \,. \end{aligned} \tag{3.4}$$

Since we have chosen $x \geq 0$, we can apply Jordan's lemma. On the arc C_R, where $z = Re^{ix\theta}$ for $\theta \in [0, \pi]$, we have

$$\left| \frac{dz}{z^2 - (2\lambda - 1)} \right| = \left| \frac{Rie^{ix\theta} d\theta}{R^2 e^{2xi\theta} - (2\lambda - 1)} \right| \leq \frac{R d\theta}{R^2 - (2\lambda - 1)} \to 0 \quad \text{as } R \to \infty \,. \tag{3.5}$$

Thus we deduce from Jordan's Lemma that

$$\lim_{R \to \infty} \int_{C_R} e^{ixz} \frac{1}{z^2 - (2\lambda - 1)} dz = 0 \,. \tag{3.6}$$

Then, from Limiting Contours IV we have for the semi-circular contributions (in the clockwise direction hence picking up a negative)

$$\lim_{r_1 \to 0} \int_{C_{r_1}} \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz = -i\pi \operatorname*{Res}_{z = z_-} \{g(z)\} = -i\pi \frac{e^{ixz_-}}{z_- - z_+} = \frac{i\pi}{2} \frac{e^{-ix\sqrt{2\lambda - 1}}}{\sqrt{2\lambda - 1}} \,, \tag{3.7}$$

and similarly
$$\lim_{r_2 \to 0} \int_{C_{r_2}} \frac{e^{ixz}}{z^2 - (2\lambda - 1)} dz = -\frac{i\pi}{2} \frac{e^{ix\sqrt{2\lambda - 1}}}{\sqrt{2\lambda - 1}} \,. \tag{3.8}$$

Taking the limits $R \to \infty$, $r_1, r_2 \to 0$ we see the first term in (3.4) is merely the principal value of the integral we want to take in (3.2). Collecting all of this into (3.4) we calculate for $x \geq 0$

$$\begin{aligned} \mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ixk}}{k^2-(2\lambda-1)}dk &= \frac{1}{\pi}\frac{i\pi}{2\sqrt{2\lambda-1}}\left(e^{ix\sqrt{2\lambda-1}} - e^{-ix\sqrt{2\lambda-1}}\right) \\ &= -\frac{1}{\sqrt{2\lambda-1}}\sin(\sqrt{2\lambda-1}\,x)\,. \end{aligned} \tag{3.9}$$

We can then perform the same procedure with minor tweaks for the case $x < 0$. This time we analyse the same integral as in (3.3) by closing contour in the *lower half plane* in the *clockwise* direction. Observing the analogous equation in (3.4) we see that the direction of the part on the real axis remains the same. The residual contributions will simply be (with positive [anti-clockwise] orientation this time)

$$\lim_{r_1\to 0}\int_{C_{r_1}}\frac{e^{ixz}}{z^2-(2\lambda-1)}dz = -\frac{i\pi}{2}\frac{e^{-ix\sqrt{2\lambda-1}}}{\sqrt{2\lambda-1}} \quad \text{and} \tag{3.10}$$

$$\lim_{r_2\to 0}\int_{C_{r_2}}\frac{e^{ixz}}{z^2-(2\lambda-1)}dz = \frac{i\pi}{2}\frac{e^{ix\sqrt{2\lambda-1}}}{\sqrt{2\lambda-1}}\,. \tag{3.11}$$

Then, since our contour is now in the lower half-plane, Jordan's lemma gives us

$$\lim_{R\to\infty}\int_{C_R} e^{ixz}\frac{1}{z^2-(2\lambda-1)}dz = \lim_{R\to\infty}\int_{C_R} e^{-i|x|z}\frac{1}{z^2-(2\lambda-1)}dz = 0\,. \tag{3.12}$$

Hence the only change that has occurred is the change of sign in the residuals, meaning for $x < 0$ we now have (where we move the negative into the odd sin to help determine the final result)

$$\mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ixk}}{k^2-(2\lambda-1)}dk = -\frac{1}{\sqrt{2\lambda-1}}\sin(-\sqrt{2\lambda-1}\,x)\,. \tag{3.13}$$

Combining the two results then gives us

$$\boxed{\mathrm{PV}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ixk}}{k^2-(2\lambda-1)}dk = -\frac{1}{\sqrt{2\lambda-1}}\sin(\sqrt{2\lambda-1}\,|x|)}\,. \tag{3.14}$$

Part b)

We can then verify that our answer in (3.9) satisfies the integral equation in (3.3). We first note that in both cases of x, for $x - y > 0$ our domain will be $-\infty < y < x$, where $y - x > 0$ gives us $x < y < \infty$.

Case 1: $x < 0$

$$
\begin{aligned}
&\lambda \int_{-\infty}^{\infty} e^{-|x-y|} \left(-\frac{1}{\sqrt{2\lambda-1}} \right) \sin(\sqrt{2\lambda-1}\,|y|) dy \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ \int_{-\infty}^{x} e^{y-x} e^{\sqrt{2\lambda-1} i|y|} dy + \int_{x}^{\infty} e^{x-y} e^{\sqrt{2\lambda-1} i|y|} dy \right\} \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ e^{-x} \int_{-\infty}^{x} e^{(1-\sqrt{2\lambda-1}i)y} dy + e^{x} \left(\int_{x}^{0} e^{(-1-\sqrt{2\lambda-1}i)y} dy + \int_{0}^{\infty} e^{(-1+\sqrt{2\lambda-1}i)y} dy \right) \right\} \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ \frac{e^{-x}}{(1-\sqrt{2\lambda-1}i)} \left[e^{(1-\sqrt{2\lambda-1}i)y} \right]_{-\infty}^{x} - \frac{e^{x}}{(1+\sqrt{2\lambda-1}i)} \left[e^{-(1+\sqrt{2\lambda-1}i)y} \right]_{x}^{0} \right. \\
&\qquad\qquad \left. + \frac{e^{x}}{(-1+\sqrt{2\lambda-1}i)} \left[e^{(-1+\sqrt{2\lambda-1}i)y} \right]_{0}^{\infty} \right\} \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ \frac{e^{-\sqrt{2\lambda-1}ix}}{1-\sqrt{2\lambda-1}i} - \frac{e^{x}}{1+\sqrt{2\lambda-1}i} + \frac{e^{-\sqrt{2\lambda-1}ix}}{1+\sqrt{2\lambda-1}i} - \frac{e^{x}}{-1+\sqrt{2\lambda-1}i} \right\} \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ e^{-\sqrt{2\lambda-1}ix} \left(\frac{1}{1-\sqrt{2\lambda-1}i} + \frac{1}{1+\sqrt{2\lambda-1}i} \right) \right. \\
&\qquad\qquad \left. - e^{x} \left(\frac{1}{1+\sqrt{2\lambda-1}i} + \frac{1}{-1+\sqrt{2\lambda-1}i} \right) \right\} \\
&= -\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ \frac{2}{2\lambda} e^{-\sqrt{2\lambda-1}ix} + \frac{2\sqrt{2\lambda-1}i}{2\lambda} e^{x} \right\} \\
&= -\frac{\sin(-\sqrt{2\lambda-1}x)}{\sqrt{2\lambda-1}} - e^{x} \, .
\end{aligned} \tag{3.15}
$$

Throughout the calculation we have used the fact that $\lim\limits_{y \to -\infty} e^{(1+\sqrt{2\lambda-1}i)y} = 0$ due to the decay in the real part. Since we have $x < 0$, we note that $|x| = -x$. Hence putting this all together into our integral equation we have

$$
e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy = e^{x} - \frac{\sin(\sqrt{2\lambda-1}\,|x|)}{\sqrt{2\lambda-1}} - e^{x} = -\frac{\sin(\sqrt{2\lambda-1}\,|x|)}{\sqrt{2\lambda-1}} = f(x) \, , \tag{3.16}
$$

thus showing that our calculated $f(x)$ in (3.14) does indeed satisfy the integral equation (3.3).

Case 2: $x \geq 0$

It would be incredibly superfluous to perform such a lengthy calculation again when there are only minor tweaks, so we will explain the key differences instead. Because $x \geq 0$ now, line 3 of (3.15) instead becomes

$$
-\frac{\lambda}{\sqrt{2\lambda-1}} \mathrm{Im} \left\{ e^{-x} \left(\int_{-\infty}^{0} e^{(1-\sqrt{2\lambda-1}i)y} dy + \int_{0}^{x} e^{(1+\sqrt{2\lambda-1}i)y} dy \right) + e^{x} \int_{x}^{\infty} e^{(-1+\sqrt{2\lambda-1}i)y} dy \right\} . \tag{3.17}
$$

We see that this time there is a e^{-x} term attached to the expanded integral as opposed to the e^{x} in the first case - this is the main difference that carries through in the calculations. Performing the exact same steps as before we arrive at

$$\lambda \int_{-\infty}^{\infty} e^{-|x-y|} \left(-\frac{1}{\sqrt{2\lambda - 1}} \right) \sin(\sqrt{2\lambda - 1}\,|y|) dy = -\frac{\sin(\sqrt{2\lambda - 1}x)}{\sqrt{2\lambda - 1}} - e^{-x} \,, \quad (3.18)$$

and then because we now have $|x| = x$ since x is positive, we see that once again $f(x)$ satisfies the integral equation. □

Part c)

Returning to (3.14), we can rewrite $z_0 = \sqrt{2\lambda - 1}$ and take the limit $\lambda \to \frac{1}{2}+$, i.e. $z_0 \to 0+$, as

$$\lim_{z_0 \to 0+} f(x) = \lim_{z_0 \to 0+} -\frac{\sin(z_0|x|)}{z_0} = \lim_{z_0 \to 0+} -|x| \frac{\sin(z_0|x|)}{z_0\,|x|} = -|x| \,, \quad (3.19)$$

where we used the standard limit $\sin(k)/k \to 1$ as $k \to 0$. We can then verify this satisfies the integral equation, first by assuming that $x < 0$:

$$\begin{aligned}
\lambda \int_{-\infty}^{\infty} e^{-|x-y|} (-|y|) dy &= -\lambda \left(\int_{-\infty}^{x} e^{y-x} |y| dy + \int_{x}^{\infty} e^{x-y} |y| dy \right) \\
&= -\lambda \left(-e^{-x} \int_{-\infty}^{x} e^{y} y dy - e^{x} \int_{x}^{0} e^{-y} y dy + e^{x} \int_{0}^{\infty} e^{-y} y dy \right) \\
&= -\lambda \left(-e^{-x} \left[e^{y}(y-1) \right]_{-\infty}^{x} + e^{x} \left[e^{-y}(y+1) \right]_{x}^{0} - e^{x} \left[e^{-y}(y+1) \right]_{0}^{\infty} \right) \\
&= -\frac{1}{2} \left(1 - x + e^{x} - (x+1) + e^{x} \right) \\
&= x - e^{x} = -|x| - e^{-|x|} \,.
\end{aligned}$$

Thus in letting $\lambda = 1/2$ we see that this once again satisfies the integral equation. Again, the $x > 0$ is identical and will instead produce $-x - e^{-x} = -|x| - e^{-|x|}$. □

Part d)

In the lecture we had, for $k < 1/2$,

$$f_0(x) = \frac{1}{\sqrt{1 - 2\lambda}} e^{-|x|\sqrt{1-2\lambda}} \,. \quad (3.20)$$

We can calculate

$$\lim_{z_0 \to 0-} \frac{1}{z_0} e^{-z_0\,|x|} = \lim_{z_0 \to 0-} \frac{1}{z_0} - |x| + \frac{z_0 |x|^2}{2} - \frac{z_0^2 |x|^3}{6} + \cdots = -\infty \,, \quad (3.21)$$

hence showing this one-sided limit clearly does not agree with the solution in (3.19) since this limit does not exist, showing us that we really do need a full description of λ-dependent behaviour before attempting to calculate "uglier" points. We note that if we had instead $f_0(x) = \frac{1}{\sqrt{1-2\lambda}}(e^{-|x|\sqrt{1-2\lambda}} - 1)$ then this limit would agree, but this would not have yielded a solution to the integral equation.

Q4. An all order asymptotic expansion

Let $n \in \mathbb{N}$. We will evaluate the large-t behaviour of solutions the initial value problem

$$\dot{y} + y = \frac{1}{t^n} \quad \text{for} \quad t \geq 1 \quad \text{with} \quad y(1) = 0\,. \tag{4.1}$$

Part a)

We can use the integrating factor $I = e^{\int 1 dt} = e^t$ to calculate

$$\begin{aligned} & \dot{y} + y = \frac{1}{t^n}\,, \\ \implies \quad & e^t\dot{y} + e^t y = e^t t^{-n}\,, \\ \implies \quad & \frac{d}{dt}(e^t y) = e^t t^{-n}\,, \\ \implies \quad & e^t y = y(1) + \int_1^t x^{-n} e^x dx\,, \end{aligned}$$

which leads to a final implicit solution

$$\boxed{y(t) = e^{-t}\int_1^t x^{-n} e^x dx}\,. \tag{4.2}$$

Part b)

We will find the all order asymptotic expansion of the solution as $t \to \infty$ using integration by parts on the integral in (4.2) to produce a recurrence relation. Letting $u = x^{-n}$ and $v' = e^x$ we have

$$\begin{aligned} \int_1^t x^{-n} e^x dx &= \left[x^{-n}e^x\right]_1^t + n\int_1^t x^{-(n+1)} e^x dx \\ &= \left(t^{-n}e^t - e\right) + n\left(t^{-(n+1)}e^t - e + (n+1)\int_1^t x^{-(n+2)}e^x dx\right) \\ &= t^{-n}e^t\left[1 + n\frac{1}{t} + n(n+1)\frac{1}{t^2} + \dots\right] - e\left[1 + n + n(n+1) + \dots\right] \\ &= t^{-n}e^t \sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!}t^{-k} + C\,, \end{aligned} \tag{4.3}$$

where we labelled the extraneous right hand series with C. When we multiply this integral by the pre-integral term e^{-t} in (4.2) we see that the term $Ce^{-t} \to 0$ as $t \to \infty$ and importantly it does this much quicker than any t^{-m} for large t values. Hence we can return to (4.2) and write our all order asymptotic expansion as $t \to \infty$

$$\boxed{y(t) = \sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!}\frac{1}{t^{n+k}}} \tag{4.4}$$

Part c)

We can verify the asymptotic expansion $y(t) = \sum a_k$ in (4.4) is not convergent by performing the ratio test. We calculate

$$\begin{aligned} L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{(n+(k+1)-1)!}{(n-1)!\, t^{n+(k+1)}} \frac{(n-1)!\, t^{n+k}}{(n+k-1)!} \right| \\ &= \lim_{k\to\infty} (n+k)\frac{1}{t} = \infty\,, \end{aligned} \tag{4.5}$$

hence showing that the asymptotic diverges since $L > 1$ for any value of t.

Q5. Asymptotic expansion of binomial coefficients

Part a)

We will first prove the identity for the two integers $0 \leq m \leq n$

$$\binom{n}{m} = \frac{1}{2\pi i} \oint \frac{(1+z)^n}{z^{m+1}} dz \,. \tag{5.1}$$

The integrand $f(z)$ has one pole of order $m+1$ at $z=0$, hence we consider a contour on the anti-clockwise unit circle $|z|=1$. Then by the residue theorem we have

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{(1+z)^n}{z^{m+1}} dz &= \operatorname*{Res}_{z=0}\{f(z)\} \\
&= \lim_{z\to 0} \frac{1}{m!} \frac{d^m}{dz^m} \left[z^{m+1} \frac{(1+z)^n}{z^{m+1}} \right] \\
&= \lim_{z\to 0} \frac{1}{m!} \frac{d^m}{dz^m} \left[\sum_{k=0}^{n} \binom{n}{k} z^k \right] \\
&= \lim_{z\to 0} \frac{1}{m!} \left[\sum_{k=m}^{n} \binom{n}{k} k(k-1)\ldots(k-(m-1)) z^{k-m} \right] \\
&= \lim_{z\to 0} \frac{1}{m!} \left[\binom{n}{m} m! + z \sum_{k=m+1}^{n} \binom{n}{k} \frac{k!}{(k-m)!} z^{k-m-1} \right] \\
&= \binom{n}{m} \,.
\end{aligned} \tag{5.2}$$

Part b)

We now want to analyse the first two leading order terms in the asymptotic expansion of $\binom{2n}{n}$ as $n \to \infty$. Using the identity in (5.1), we can write

$$\binom{2n}{n} = \frac{1}{2\pi i} \oint \frac{(1+z)^{2n}}{z^{n+1}} dz = \frac{1}{2\pi i} \oint \frac{dz}{z} \exp\left[n\left(2\log(1+z) - \log(z)\right)\right] \,. \tag{5.3}$$

To perform our saddle point analysis, we will first define

$$g(z) = \frac{1}{z} \qquad \text{and} \qquad h(z) = 2\log(1+z) - \log(z)\,. \tag{5.4}$$

We can then calculate the saddle point of $h(z)$,

$$h'(z) = \frac{2}{1+z} - \frac{1}{z} = 0\,, \qquad \text{so} \quad z_0 = 1\,. \tag{5.5}$$

We then note that the integrand is holomorphic in a neighbourhood of this saddle point (its only singularity is at $z=0$), hence allowing us to perform the saddle point analysis around $z_0 = 1$. Noting that we have

$$h''(z) = \frac{1}{z^2} - \frac{2}{(1+z)^2}\,, \qquad \text{so} \quad h''(z_0) = \frac{1}{2} \tag{5.6}$$

we can then expand $h(z)$ about $z_0 = 1$ as follows:

$$\begin{aligned} h(z) &= h(z_0) + h'(z_0)(z - z_0) + \frac{h''(z_0)}{2}(z - z_0)^2 + \dots \\ &= 2\log(2) + \frac{1}{4}(z-1)^2 + \dots \end{aligned}$$

We can then make a change of variables $z = 1 + re^{i\theta}$ which leads to

$$h(z) = 2\log(2) + \frac{1}{4}r^2 e^{2i\theta} + \dots \tag{5.7}$$

The path of steepest descent, Γ_s, will occur when $e^{2i\theta} = -1$, so $\theta = \frac{\pi}{2}$. Hence, the contour of steepest descent passes through $z = 1$ in the positive direction parallel to the imaginary axis, so we let $z = 1 + ri$ with $dz = idr$. Before expanding more fully, we first calculate higher term expansions of $g(z)$ and $h(z)$,

$$g(r) = \frac{1}{1+ri} = 1 - ir - r^2 + \dots, \tag{5.8}$$

$$nh(r) = 2n\log(2) - \frac{n}{4}r^2 + \frac{n}{4}ir^3 + \frac{7n}{32}r^4 \dots. \tag{5.9}$$

In our integral calculation we will make the substitution $r = \frac{2}{\sqrt{n}}u$, which tells us how many terms to expand to above. In the $g(r)$ case we expand to r^2 as this will yield $1/n$. In $h(r)$, we expand to the nr^4 term as this will yield $n/n^2 = 1/n$. Hence we can now use Laplace's method and find our asymptotic expansion as follows:

$$\begin{aligned} \frac{1}{2\pi i}\oint g(z)\exp[nf(z)]dz &\sim \frac{1}{2\pi i}\exp[2n\log(2)]\int_{-\infty}^{\infty} i\,dr g(r)\exp\left[-\frac{n}{4}r^2 + \frac{n}{4}ir^3 + \frac{7n}{32}r^4 + \dots\right] \\ &\sim \frac{2^{2n}}{2\pi}\int_{-\infty}^{\infty} dr\, e^{-\frac{n}{4}r^2}\left[1 - ir - r^2 + \dots\right] e^{\frac{n}{4}ir^3 + \frac{7n}{32}r^4} \\ &= \frac{2^{2n}}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\sqrt{n}}du\, e^{-u^2}\left[1 - \frac{2}{\sqrt{n}}iu - \frac{4}{n}u^2 + \dots\right] e^{\frac{2}{\sqrt{n}}iu^3 + \frac{7}{2n}u^4} \\ &\sim \frac{2^{2n}}{\pi\sqrt{n}}\int_{-\infty}^{\infty} du\, e^{-u^2}\left[1 - \frac{2}{\sqrt{n}}iu - \frac{4}{n}u^2 + \dots\right]\left[1 + \left(\frac{2}{\sqrt{n}}iu^3 + \frac{7}{2n}u^4\right)\right. \\ &\qquad \left. + \frac{1}{2!}\left(\frac{2}{\sqrt{n}}iu^3 + \frac{7}{2n}u^4\right)^2 + \dots\right] \\ &\sim \frac{2^{2n}}{\pi\sqrt{n}}\int_{-\infty}^{\infty} du\, e^{-u^2}\left[1 - \frac{2i}{\sqrt{n}}u - \frac{4}{n}u^2 + ..\right]\left[1 + \frac{2i}{\sqrt{n}}u^3 + \frac{7}{2n}u^4 - \frac{2}{n}u^6 + ..\right] \\ &= \frac{2^{2n}}{\pi\sqrt{n}}\int_{-\infty}^{\infty} du\, e^{-u^2}\left[1 - \frac{2i}{\sqrt{n}}u - \frac{4}{n}u^2 + \frac{2i}{\sqrt{n}}u^3 + \frac{4}{n}u^4 - \frac{8i}{n^{3/2}}u^5\right. \\ &\qquad \left. + \frac{7}{2n}u^4 - \frac{14i}{n^{3/2}}u^5 - \frac{14}{n^2}u^6 - \frac{2}{n}u^6 + \frac{4i}{n^{3/2}}u^7 + \frac{8}{n^2}u^8 + ..\right] \\ &= \frac{2^{2n}}{\pi\sqrt{n}}\int_{-\infty}^{\infty} du\, e^{-u^2}\left[1 - \frac{4}{n}u^2 + \frac{15}{2n}u^4 - \frac{2}{n}u^6 + \dots\right] \\ &= \frac{2^{2n}}{\pi\sqrt{n}}\left[\sqrt{\pi} - \frac{\sqrt{\pi}}{8n} + \dots\right]. \end{aligned}$$

In the fifth line we only needed the $(u^3)^2$ term from the quadratic expansion as this is the only part that contributes to $1/n$, so we omit expanding the rest. In the seventh line we used the fact that $\int_{\mathbb{R}} r^{2k+1} e^{-r^2} dr = 0$ for any integer k, hence allowing us to remove the odd contributions. We also discarded the higher order $1/n^2$ terms along the way. Hence we arrive at the final solution,

$$\boxed{\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left[1 - \frac{1}{8n} + \dots\right] \quad \text{as } n \to \infty} \ . \tag{5.10}$$

□