Advanced Methods Differential Equations Assignment 1

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Q1. Fredholm Alternative application

Consider the boundary problem for $\alpha, \beta \in \mathbb{R}$

$$Lu \equiv \left(\frac{d^2}{dx^2} + 25\right) u(x) = \alpha x^{2021} \sin(3x) + \beta \sin(5x) \equiv f(x)\,, \tag{1.1}$$
$$\text{with } u(-\pi) = 0,\ u(\pi) = 0, \quad \text{for } -\pi < x < \pi\,.$$

Note that it is trivial that L is self-adjoint, and these are homogeneous boundary conditions, hence we may apply the Fredholm Alternative to our analysis. We first analyse the homogeneous problem $u'' + 25u = 0$ which leads to a solution

$$u_1(x) = A\cos(5x) + B\sin(5x) = B\sin(5x)\,, \tag{1.2}$$

where the second equality follows from a simple application of the boundary conditions in (1.1) - note, however, that these boundary conditions are not sufficient to specify the value of the arbitrary constant B.

If we had an additional boundary condition that ensured $B = 0$, giving only one (trivial) solution to homogeneous problem, namely $u_1(x) = 0$, then the Fredholm Alternative tells us that $Lu = f$ has a *unique* solution for any values of α, β.

Supposing that $B \neq 0$, we have a nontrivial solution $u_1(x)$ to the homogeneous problem $Lu = 0$. Thus to determine the number of solutions to $Lu = f$ we want to analyse $\langle f(x), u_1(x)\rangle$. We can then perform the straightforward calculation

$$\begin{aligned}
\langle f(x), u_1(x)\rangle &= \int_{-\pi}^{\pi} \Big(\alpha x^{2021}\sin(3x) + \beta\sin(5x)\Big) B\sin(5x)dx \\
&= \alpha B \int_{-\pi}^{\pi} x^{2021}\sin(3x)\sin(5x)dx + \beta B\int_{-\pi}^{\pi}\sin^2(5x)dx \\
&= \frac{\beta B}{10}\int_{-5\pi}^{5\pi}\big(1-\cos(2w)\big)\,dw = \frac{\beta B}{10}\left[w - \frac{1}{2}\sin(2w)\right]_{-5\pi}^{5\pi} = \beta B\pi\,.
\end{aligned} \tag{1.3}$$

Note that in the third line we used the fact that $g(x) = x^{2021}\sin(3x)\sin(5x)$ is a product of three odd functions, hence is odd itself, and so $\int_{-\pi}^{\pi} g(x)dx = 0$ due to the symmetric domain. Therefore since $\langle f, u_1\rangle = \beta B\pi$, by the Fredholm Alternative we see that

$$Lu = f \quad \text{has} \quad \begin{cases} \text{no solution} & \text{if } \beta \neq 0 \\ \text{infinitely many solutions} & \text{if } \beta = 0 \end{cases}. \tag{1.4}$$

Note that we have shown that the value of α has no bearing on the number of solutions to the boundary problem. $\square$

Q2. Application of the Frobenius method

Consider the second order differential equation

$$(x-\beta)y'' - xy' + \gamma y = 0\,, \tag{2.1}$$

where β, γ are parameters. We can rewrite this equation into the standard Frobenius form

$$y'' + \frac{p(x)}{(x-\beta)}y' + \frac{q(x)}{(x-\beta)^2}y := y'' - \frac{x}{(x-\beta)}y' + \frac{\gamma(x-\beta)}{(x-\beta)^2}y = 0\,. \tag{2.2}$$

Part a)

With $p(x) = -x$ and $q(x) = \gamma(x-\beta)$ as above, we see that both of these simple linear functions are clearly analytic at the point $x = \beta$, hence $x = \beta$ is a regular singular point of this differential equation. In other words, in writing $a_2 y'' + a_1 y' + a_0 y = 0$ with $a_2(x) = 1$, $a_1(x) = -\frac{x}{x-\beta}$ and $a_0(x) = \frac{\gamma}{x-\beta}$, we see that $a_1(x)$ and $a_0(x)$ are both singular at $x = \beta$, but $(x-\beta)a_1(x)$ and $(x-\beta)^2 a_0(x)$ are both analytic at $x = \beta$, hence it is a regular singular point.

Part b)

Given that $x = \beta$ is a regular singular point, we thus want to analyse a series of the form

$$y(x) = (x-\beta)^\alpha A(x) = \sum_{n=0}^{\infty} a_n (x-\beta)^{\alpha+n}\,. \tag{2.3}$$

We can first make a simple transformation $z = x - \beta$ (which induces no change in differentials, i.e. $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$) and hence analyse

$$y'' - \frac{z+\beta}{z}y' + \frac{\gamma z}{z^2}y = 0 \qquad \text{with} \quad y(z) = \sum_{n=0}^{\infty} a_n z^{\alpha+n} \tag{2.4}$$

about the point $z = 0$. Our two coefficient functions have Taylor expansions of the form

$$p(z) = p_0 + p_1 z := -\beta - z\,, \quad \text{and} \quad q(x) = q_0 + q_1 z := 0 + \gamma z\,. \tag{2.5}$$

Using the Frobenius method as outlined in the Week 4 lecture notes, we know that the leading term of our series, i.e. the z^α term, can be found by solving the indicial equation

$$P(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0 = \alpha\big(\alpha - (\beta+1)\big) = 0\,, \tag{2.6}$$

hence giving the two indices of $\alpha_1 = \beta + 1$ and $\alpha_2 = 0$. Note here that this labelling to ensure that $\mathrm{Re}(\alpha_1) \geq \mathrm{Re}(\alpha_2)$ clearly only holds if $\beta \geq -1$, otherwise we will need to alter it. In any case, the first solution $y_1(z)$ associated to α_1 is guaranteed to be analytic in a neighbourhood of $z = 0$, meaning we can write

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha_1}\,. \tag{2.7}$$

The analyticity of the second series solution is highly dependent on the parameter β. Suppose first that $\beta \geq -1$ to ensure our labelling as above. If $\beta \neq -1, 0, 1, 2, \ldots$ then the second series will be of the same Frobenius form as in (2.3), suitably adjusted with different coefficients, and analytic within its radius of convergence including the point $z = 0$. In the case of $\beta = -1$, the second solution will have the form of (2.13) and hence not be analytic at $z = 0$ due to the presence of the log term - see part c) for details. If $\beta = -1, 0, 1, 2, \ldots$, then either of these aforementioned solutions are possible - it then depends on the precise differential equation in question.

If $\beta < -1$, then we relabel the α's to be $\alpha_1 = 0$ and $\alpha_2 = \beta + 1$ so that $\mathrm{Re}(\alpha_1) \geq \mathrm{Re}(\alpha_2)$, hence $\alpha_1 - \alpha_2 = -\beta - 1$. So, if $-\beta - 1 \neq 0, 1, 2, \ldots$, that is $\beta \neq -1, -2, -3, \ldots$, then we will again have a second series solution of the form (2.3) and so the same analyticity properties hold. If $-\beta - 1 = 0$, so $\beta = -1$, then we are in the same case as before and will have a non-analyticity at $z = 0$. If $-\beta - 1 = 1, 2, 3, \ldots$, so $\beta = -2, -3, -4, \ldots$ then we may be in either case, again dependent on the precise differential equation.

Part c)

Consider the case where $\beta = -1$ and $\gamma \neq 0, 1, 2, 3 \ldots$. This leads to a double root of $P(\alpha)$ where $\alpha_1 = \alpha_2 = 0$ which will have bearing on the second solution. We start by analysing the first solution of the form $y_1(x; \alpha_1) = \sum_{n=0}^{\infty} a_n z^n$. Substituting this into (2.4) we find

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} a_n n(n-1) z^{n-2} - (1 + \beta z^{-1}) \sum_{n=1}^{\infty} a_n n z^{n-1} + \gamma z^{-1} \sum_{n=0}^{\infty} a_n z^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1) z^{n-2} - \sum_{n=1}^{\infty} a_n n z^{n-1} - \beta \sum_{n=1}^{\infty} a_n n z^{n-2} + \gamma z^{-1} \sum_{n=0}^{\infty} a_n z^{n-1} \qquad (2.8) \\
&= (\gamma a_0 - \beta a_1) z^{-1} + \sum_{n=0}^{\infty} \Big(a_{n+2}(n+1)(n+2) - a_{n+1}(n+1) - \beta a_{n+2}(n+2) + \gamma a_{n+1} \Big) z^n .
\end{aligned}$$

A power series is identically 0 if and only if all coefficients are 0, therefore we can solve for a_n by solving when each coefficient is 0. We first see that, for a free parameter a_0 we have the following recurrence relation

$$a_1 = \frac{-\gamma a_0}{-\beta}, \qquad \text{and} \quad a_{n+1} = \frac{n - \gamma}{(n+1)(n-\beta)} a_n \quad \text{for } n \geq 1, \qquad (2.9)$$

where the first few terms are

$$a_2 = \frac{(1-\gamma)}{2(1-\beta)} \frac{-\gamma}{-\beta} a_0, \quad a_3 = \frac{(2-\gamma)}{3(2-\beta)} \frac{(1-\gamma)}{2(1-\beta)} \frac{-\gamma}{-\beta} a_0, \quad a_4 = \frac{(3-\gamma)}{4(3-\beta)} \frac{(2-\gamma)}{3(2-\beta)} \frac{(1-\gamma)}{2(1-\beta)} \frac{-\gamma}{-\beta} a_0 .$$

Putting this all together we can find the form of a_n, namely

$$a_n = \frac{((n-1)-\gamma)((n-2)-\gamma)\ldots(1-\gamma)(-\gamma)}{n!((n-1)-\beta)((n-2)-\beta)\ldots(1-\beta)(-\beta)} a_0 \quad \text{for } n \geq 1 . \qquad (2.10)$$

We can then appeal to Pochammer notation to see that we can write $a_n = \frac{(-\gamma)_n}{n!(-\beta)_n} a_0$. But then since $\beta = -1$ we have $(-\beta)_n = (1)_n = n!$, so we may finally write our first series solution as

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{(-\gamma)_n}{(n!)^2} a_0 . \qquad (2.11)$$

To analyse the analyticity we may appeal to a radius of convergence calculation using the ratio test. Recall that the radius of convergence of $y_1(z)$ is those values of z such that $\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| < 1$ to ensure the series converges.

Thus we can calculate

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty}\left|\frac{(-\gamma)_{n+1}a_0}{((n+1)!)^2}\frac{(n!)^2}{(-\gamma)_n a_0}z\right| = \lim_{n\to\infty}\left|\frac{(n-\gamma)}{(n+1)^2}z\right| ,$$

$$\text{hence we require} \quad |z| < \lim_{n\to\infty}\left|\frac{(n+1)^2}{(n-\gamma)}\right| = \infty \tag{2.12}$$

and so the radius of convergence is infinite, hence $y_1(z)$ is analytic everywhere.

Due to the aforementioned double root of $P(\alpha)$ for $\beta = -1$, we know by the Frobenius method from lectures that the second linearly independent series must have the form

$$y_2(z) = y_1(z)\log(z) + \sum_{n=0}^{\infty} c_n z^n , \tag{2.13}$$

where we have used the fact that $z^{n+\alpha} = z^n$ since $\alpha = 0$. Clearly due to the log term this solution is not analytic at $z = x - \beta = 0$, but it is analytic in a neighbourhood (only for $z > 0$ it seems). Note that our condition $\gamma \neq 0, 1, 2, 3, \ldots$ ensures that $a_n \neq 0$ for any n, but this situation is not much of an issue anyway, as we will see in part d).

Part d)

Suppose $\gamma = m$ for some $m = 0, 1, 2, \ldots$ and still $\beta = -1$. Then the series solution will have a finite number of terms because for each $n > \gamma$ we have $a_n \propto (\gamma - n)$, hence giving $a_n = 0$. If we take $\gamma = 2$ then the analytic solution looks like

$$y_1(z) = a_0 + \frac{2}{\beta}a_0 z + \frac{2}{2(\beta-1)\beta}a_0 z^2 + 0z^3 + 0z^4 + 0z^5 \ldots .$$

Using the initial condition $y(0) = 1$, we thus have

$$y_1(z) = 1 - 2z + \frac{1}{2}z^2 . \tag{2.14}$$

Q3. Sturm-Liouville example

Consider the Sturm-Liouville problem

$$(x+1)y'' - xy' + \lambda y = 0\,, \quad 0 < x < 1\,, \quad y(0) = 0\,,\; y(1) = 0\,.$$

We note that this is precisely the problem we have considered in Q2 with $\beta = -1$, and instead of considering the parameter γ we now consider the eigenvalue λ. Again, we make the transformation $z = x - \beta = x + 1$ so that our problem is now

$$y'' + \left(\frac{1}{z} - 1\right) y' + \frac{\lambda}{z} y = 0\,, \quad 1 < z < 2\,, \quad y(1) = 0\,,\; y(2) = 0\,. \tag{3.1}$$

Part a)

To determine information about the eigenvalues λ, we first want to put (3.1) into Sturm-Liouville form, that is,

$$\frac{d}{dz}\left(p(z)\frac{dy}{dz}\right) + \big(q(z) + \lambda r(z)\big)y = 0 \tag{3.2}$$

for a finite interval $a \le z \le b$ with $p(z) > 0$ and $r(z) > 0$. To do this, we can use an integrating factor to collapse the first two terms of (3.1) into a derivative of a product, namely

$$\mu = \exp\left(\int (z^{-1} - 1)dz\right) = \exp(\log(z) - z) = ze^{-z}\,. \tag{3.3}$$

We can then multiply both sides by μ,

$$ze^{-z}y'' + ze^{-z}\left(\frac{1}{z} - 1\right) y' + ze^{-z}\frac{\lambda}{z} y = 0\,,$$

which thus simplifies into Sturm-Liouville form,

$$\frac{d}{dz}\left(ze^{-z}\frac{dy}{dz}\right) + \lambda e^{-z} y = 0\,. \tag{3.4}$$

Hence we can identify this with (3.2) by writing

$$p(z) = ze^{-z}\,, \quad q(z) = 0\,, \quad r(z) = e^{-z}\,. \tag{3.5}$$

We then see that on our finite interval $[a, b] = [1, 2]$, none of $p(1)\,, p(2)\,, r(1)\,, r(2)$ vanish or diverge, hence we conclude that this is a *regular* Sturm-Liouville (S-L) problem.

As such, the theorem from lectures tells us that all eigenvalues λ of a regular S-L problem must be real. Furthermore, in rewriting our Dirichlet boundary conditions as

$$\begin{aligned} &\alpha y(1) + \zeta y'(1) := 1y(1) + 0y'(1) = 0 \\ \text{and} \quad &\gamma y(2) + \delta y'(2) := 1y(2) + 0y'(2) = 0\,, \end{aligned}$$

we see that $q(z) = 0$, $\alpha \cdot \zeta = 0 \le 0$ and $\gamma \cdot \delta = 0 \ge 0$, hence the eigenvalues λ are strictly non-negative. Therefore we have $\lambda \in (0, \infty)$ for all eigenvalues.

Part b)

Let $y_k(z)$ and λ_k denote the eigenfunctions and corresponding eigenvalues associated to the Sturm-Liouville operator L as above, that is, $Ly = \lambda r(z)y(z)$. Suppose we are given an arbitrary function $f(z)$ that is "sufficiently nice", then we can hope to express it in terms of the eigenfunctions of L, that is, write

$$f(z) = \sum_{n=0} c_n y_k(z) \quad \text{for some coefficients} \quad c_n \in \mathbb{R}\,. \tag{3.6}$$

We know by the theorem in lectures that the eigenfunctions of L are orthogonal with respect to the weight function $r(z)$, that is,

$$\langle ry_n, y_k\rangle = \langle r(z)y_n(z), y_k(z)\rangle = \int_1^2 e^{-z} y_n(z)y_k(z)dz = 0 \quad \text{for} \quad n \neq k\,. \tag{3.7}$$

By the self-adjointness of L we also have

$$\langle y_k, f\rangle = \langle y_k, Ly\rangle = \langle Ly_k, y\rangle = \langle \lambda_k r y_k, y\rangle\,. \tag{3.8}$$

Hence, using our expression in (3.6), we can write for some fixed k index

$$\langle y_k, f\rangle = \langle \lambda_k r y_k, f\rangle = \lambda_k \left\langle r y_k, \sum_{n=0}^{\infty} c_n y_n \right\rangle \tag{3.9}$$

$$= \lambda_k \sum_{n=0}^{\infty} \langle r y_k, c_n y_n\rangle = \lambda_k \sum_{n=0}^{\infty} c_k \langle r y_k, y_n\rangle \delta_{k,n} = \lambda_k c_k \langle r y_k, y_k\rangle\,, \tag{3.10}$$

where we have used the linearity of the inner product (whilst ignoring the important technical details about pulling the infinite sum out, see a Functional Analysis textbook for a more rigorous approach with particular conditions on $f(z)$), and the orthogonality of the y_k's (where $\delta_{k,n}$ denotes the Kronecker delta symbol). Therefore, using the standard inner product on function spaces,

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

we have the following formula for c_n in our case,

$$c_k = \frac{\langle y_k, f\rangle}{\lambda_k \langle r y_k, y_k\rangle} = \frac{\int_1^2 y_k(z) f(z)dz}{\lambda_k \int_1^2 e^{-z} y_k^2(z)dz}\,. \tag{3.11}$$

Part c)

As mentioned in the preamble, the ODE in (3.1) is precisely the ODE in (2.4) with $\beta = -1$ (as we had there) and the eigenvalues λ taking the place of the free parameter γ. However, now that we have analysed this question in the context of the Sturm-Liouville form, we see that our conclusion in part a) that $\lambda \in (0, \infty)$ now tells us that we must also have $\gamma \in (0, \infty)$ in (2.4) or else we would have trivial solutions for $y(z)$.

Furthermore, in the problem of Q3 we have now specified boundary conditions on the domain $[1, 2]$ which play an important role. With these boundary conditions, we can now say that if γ is *not* one of the eigenvalues calculated for the S-L problem, then no solution will exist for the boundary value problem. With a bit of work, one would be able to calculate these eigenvalues, but Sturm-Liouville theory at least guarantees what form they must have. Note the further point that in Q2 we expanded about the point $z = 0$, which isn't actually in the domain $[1, 2]$ in Q3.

Q4. Irregular Singular Point

We want to find the leading order behaviour of the two solutions to

$$x^6 y''(x) = y(x) \quad \text{as } x \to 0^+\,, \tag{4.1}$$

where in rearranging to $y'' = \frac{1}{x^6}y$ it is clear that $x = 0$ is an irregular singular point. To do this we may consider a trial solution of the form

$$y(x) = e^{S(x)}, \quad \text{so } y'(x) = S'(x)e^{S'(x)}, \quad \text{so } y''(x) = S''(x)e^{S(x)} + (S'(x))^2 e^{S(x)}\,, \tag{4.2}$$

for some unknown complex valued function $S(x)$. From here on out we will drop the cumbersome function notation. Using the relations generated by this trial solution, we can substitute into (4.1) to find

$$S''e^S + (S')^2 e^S - \frac{1}{x^6}e^S = \left(S'' + (S')^2 - \frac{1}{x^6}\right)e^S = 0\,,$$

$$\text{hence} \quad S'' + (S')^2 - \frac{1}{x^6} = 0 \tag{4.3}$$

since e^S will never be 0. We may then appeal to a general fact for irregular singular points (ISPs) that $S'' \ll (S')^2$ in the neighbourhood of the singularity we are considering, therefore we may simplify (4.3) to be

$$(S')^2 \sim \frac{1}{x^6}\,, \quad \text{so } S'(x) \sim \pm x^{-3}\,, \quad \text{so } S(x) \sim \mp\frac{1}{2}x^{-2} \quad \text{as } x \to 0^+\,. \tag{4.4}$$

Consider the first solution given by $S_1(x) = -\frac{1}{2}x^{-2} + C(x)$ for some function $C(x)$ such that

$$C(x) \ll -\frac{1}{2}x^{-2}\,, \text{ so } C'(x) \ll x^{-3} \text{ and } C''(x) \ll -3x^{-4} \text{ as } x \to 0^+\,. \tag{4.5}$$

To find the leading order behaviour of (4.1) we seek to understand the asymptotic form of $C(x)$, so we can start by calculating

$$S_1' = x^{-3} + C'(x)\,, \quad \text{and} \quad S_1'' = -3x^{-4} + C''(x)\,. \tag{4.6}$$

We can then plug this into (4.3) to get

$$\begin{aligned}
0 &= S_1'' + (S_1')^2 - x^{-6} \\
&= -3x^{-4} + C''(x) + (x^{-3} + C'(x))^2 - x^{-6} \\
&= -3x^{-4} + C''(x) + x^{-6} + 2x^{-3}C'(x) + C'(x)^2 - x^{-6} \\
&= C'(x)^2 + 2x^{-3}C'(x) - 3x^{-4} + C''(x)\,.
\end{aligned} \tag{4.7}$$

Using our information in (4.5), which we can append with $(C')^2 \ll x^{-3}C'$, we may write

$$C'(x)^2 + 2x^{-3}C'(x) \sim 2x^{-3}C'(x)\,, \quad \text{and} \quad C''(x) - 3x^{-4} \sim -3x^{-4}\,. \tag{4.8}$$

Hence, plugging this into (4.7) we have

$$2x^{-3}C'(x) - 3x^{-4} = 0\,, \text{ so } C'(x) = \frac{3}{2x}\,, \text{ so } C(x) = \frac{3}{2}\log x + D(x) \text{ as } x \to 0^+ \tag{4.9}$$

for some function $D(x) \ll \log x$ as $x \to 0^+$ which gives

$$S_1(x) = -\frac{1}{2}x^{-2} + \frac{3}{2}\log x + D(x)\,. \tag{4.10}$$

We can now play the same game with $D(x)$. Since $D(x) \ll \log x$ we have

$$D'(x) \ll x^{-1} \quad \text{and} \quad D''(x) \ll -x^{-2}\,, \tag{4.11}$$

and then

$$S_1' = x^{-3} + \frac{3}{2}x^{-1} + D'(x) \quad \text{and} \quad S_1'' = -3x^{-4} - \frac{3}{2}x^{-2} + D''(x)\,. \tag{4.12}$$

Substituting these into (4.3) we get

$$\begin{aligned}
0 &= S_1'' + (S_1')^2 - x^{-6} \\
&= -3x^{-4} - \frac{3}{2}x^{-2} + D''(x) + \left(x^{-3} + \frac{3}{2}x^{-1} + D'(x)\right)^2 - x^{-6} \\
&= -3x^{-4} - \frac{3}{2}x^{-2} + D''(x) + x^{-6} + 3x^{-4} + 2x^{-3}D'(x) + \frac{9}{4}x^{-2} + 3x^{-1}D'(x) + D'(x)^2 - x^{-6} \\
&= D''(x) + \frac{3}{4}x^{-2} + (3x^{-1} + 2x^{-3})D'(x) + D'(x)^2\,. \qquad (4.13)
\end{aligned}$$

From (4.11) we then have

$$D''(x) + \frac{3}{4}x^{-2} \sim \frac{3}{4}x^{-2}\,, \quad \text{and} \quad 3x^{-1}D'(x) + D'(x)^2 \sim 3x^{-1}D'(x) \quad \text{as} \quad x \to 0^+\,, \tag{4.14}$$

but then we also know that $\frac{3}{x} + \frac{2}{x^3} \sim \frac{2}{x^3}$ as $x \to 0^+$, hence we may use these simplifications to write

$$\frac{2}{x^3}D'(x) + \frac{3}{4x^2} = 0\,, \quad \text{so} \quad D(x) \sim -\frac{3}{16}x^2 + d \tag{4.15}$$

for some constant d. But since $-\frac{3}{16}x^2 \to 0$ as $x \to 0^+$ we may simply write $D(x) \sim d$ as $x \to 0^+$.

Putting all of this together, we now have the leading order solution for $y_1(x)$ as

$$y_1(x) \sim e^{S_1(x)} \sim e^{-\frac{1}{2}x^{-2} + \frac{3}{2}\log x + d} = C_1 x^{\frac{3}{2}} e^{-\frac{1}{2x^2}} \tag{4.16}$$

for some constant C_1.

For the second solution, we may simply appeal to the symmetry of the solution $S(x) \sim \mp\frac{1}{2}x^{-2}$. In all of our above calculations when we had some relation like $C(x) \ll kf(x)$ as $x \to 0^+$ for some constant $k \in \mathbb{R}$ and function $f(x)$, we could always neglect the constant k as it never changed the asymptotic behaviour in such relations, regardless of whether it was positive or negative. So, if one carries through the exact same calculations as above flipping the sign of many terms following from differentiation and such, one finds that

$$y_2(x) \sim e^{S_2(x)} \sim e^{\frac{1}{2}x^{-2} + \frac{3}{2}\log x + d} = C_2 x^{\frac{3}{2}} e^{\frac{1}{2x^2}} \tag{4.17}$$

for some constant C_2. Note that, taking limits, we see that $y_1 \to 0$ as $x \to 0^+$ whereas $y_2 \to \infty$ as $x \to 0^+$. Thus we have found the leading order behaviour of both solutions to the original ODE. □