Algebraic Geometry Assignment 2

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1 Localisation

Q3. Local ring and residue field

Let A be a ring an $\mathfrak{p} \subseteq A$ a prime ideal. Knowing that $A - \mathfrak{p} \subseteq A$ is a multiplicative set, we can define the localisation at $\mathfrak{p}$ as

$$A_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}A = \{\frac{a}{b} : a \in A\,, b \in A - \mathfrak{p}\}\,. \tag{1.3.1}$$

Part a)

We claim that $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal

$$\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}^e = \mathfrak{p}A_{\mathfrak{p}} = \{\frac{a}{b} : a \in \mathfrak{p}\,, b \in A - \mathfrak{p}\}\,. \tag{1.3.2}$$

It is clear that this is an ideal. First observe that $u/b \in A_{\mathfrak{p}}$ is a unit if and only if $u \in A - \mathfrak{p}$: the reverse implication is clear, so suppose u/b is a unit such that for $u' \in A$, $b' \in A - \mathfrak{p}$ we have $uu'/(bb') = 1$, so by definition there is some $t \in A - \mathfrak{p}$ such that $tuu' = tbb'$. Since $A - \mathfrak{p}$ is a multiplicative set, we have $tbb' \in A - \mathfrak{p}$ and if u was in $\mathfrak{p}$, then $tuu' \in \mathfrak{p}$ since $\mathfrak{p}$ is an ideal, contradicting that $tbb' \in A - \mathfrak{p}$ and so $u \in A - \mathfrak{p}$ proving the claim. Hence we see that the set of non-units in $A_{\mathfrak{p}}$ is precisely $\mathfrak{m}_{\mathfrak{p}}$.

Suppose $I \subseteq A_{\mathfrak{p}}$ is an ideal such that $\mathfrak{m}_{\mathfrak{p}} \subsetneq I$. Then I must contain a unit, meaning $I = A_{\mathfrak{p}}$ by necessity, hence $\mathfrak{m}_{\mathfrak{p}}$ is a maximal ideal. To show uniqueness, suppose $\mathfrak{m}' \subset A_{\mathfrak{p}}$ is another maximal ideal that is proper, hence must be contained in the set of non-units, i.e. $\mathfrak{m}' \subseteq \mathfrak{m}_{\mathfrak{p}}$, but since $\mathfrak{m}'$ is maximal this implies that $\mathfrak{m}' = \mathfrak{m}_{\mathfrak{p}}$, hence $\mathfrak{m}_{\mathfrak{p}}$ is unique and so $A_{\mathfrak{p}}$ is a local ring. $\square$

Part b)

We define the residue field of A at $\mathfrak{p}$ as $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. To show that $\mathrm{Frac}(A/\mathfrak{p}) \cong \kappa(\mathfrak{p})$, where

$$\mathrm{Frac}(A/\mathfrak{p}) = (A/\mathfrak{p} - \{0\})^{-1}A/\mathfrak{p} = \{\frac{x}{y} : x \in A/\mathfrak{p},\ y \in A/\mathfrak{p} - \{0\}\}\,, \tag{1.3.3}$$

we can define a map

$$\phi : \text{Frac}(A/\mathfrak{p}) \to A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} : \frac{a+\mathfrak{p}}{b+\mathfrak{p}} \mapsto \frac{a}{b} + \mathfrak{m}_\mathfrak{p} \,. \tag{1.3.4}$$

First note that the map is well defined as $a/b + \mathfrak{m}_\mathfrak{p} \in \kappa(\mathfrak{p})$ since $a/b \in A/\mathfrak{p}$, and further if $x/y = x'/y' \in \text{Frac}(A/\mathfrak{p})$, then

$$\phi\left(\frac{a+\mathfrak{p}}{b+\mathfrak{p}}\right) = \frac{a}{b} + \mathfrak{m}_\mathfrak{p} = \frac{a'}{b'} + \mathfrak{m}_\mathfrak{p} = \phi\left(\frac{a'+\mathfrak{p}}{b'+\mathfrak{p}}\right) \,. \tag{1.3.5}$$

We see this is a ring homomorphism since addition homomorphism holds,

$$\begin{aligned}\phi\left(\frac{a+\mathfrak{p}}{b+\mathfrak{p}} + \frac{a'+\mathfrak{p}}{b'+\mathfrak{p}}\right) = \phi\left(\frac{ab'+a'b+\mathfrak{p}}{bb'+\mathfrak{p}}\right) &= \frac{ab'+a'b}{bb'} + \mathfrak{m}_\mathfrak{p} \\ &= \left(\frac{a}{b} + \mathfrak{m}_\mathfrak{p}\right) + \left(\frac{a'}{b'} + \mathfrak{m}_\mathfrak{p}\right) \\ &= \phi\left(\frac{a+\mathfrak{p}}{b+\mathfrak{p}}\right) + \phi\left(\frac{a'+\mathfrak{p}}{b'+\mathfrak{p}}\right) \,, \end{aligned} \tag{1.3.6}$$

and the multiplication homomorphism holds,

$$\phi\left(\frac{a+\mathfrak{p}}{b+\mathfrak{p}}\frac{a'+\mathfrak{p}}{b'+\mathfrak{p}}\right) = \phi\left(\frac{aa'+\mathfrak{p}}{bb'+\mathfrak{p}}\right) = \left(\frac{a}{b} + \mathfrak{m}_\mathfrak{p}\right)\left(\frac{a'}{b'} + \mathfrak{m}_\mathfrak{p}\right) = \phi\left(\frac{a+\mathfrak{p}}{b+\mathfrak{p}}\right)\phi\left(\frac{a'+\mathfrak{p}}{b'+\mathfrak{p}}\right) \,,$$

and the identity maps to the identity,

$$\phi(1_{\text{Frac}(A/\mathfrak{p})}) = \phi\left(\frac{a+\mathfrak{p}}{a+\mathfrak{p}}\right) = \frac{a}{a} + \mathfrak{m}_\mathfrak{p} = 1 + \mathfrak{m}_\mathfrak{p} = 1_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}} \,. \tag{1.3.7}$$

Finally, we see that the kernel of our map is $\{0\} \subseteq \text{Frac}(A/\mathfrak{p})$, since $a/b + \mathfrak{m}_\mathfrak{p} = 0 \in \kappa(\mathfrak{p})$ if and only if $a/b \in \mathfrak{m}_\mathfrak{p}$, which is to say that $a \in \mathfrak{p}$ and $b \in A - \mathfrak{p}$. But if $a \in \mathfrak{p}$, then $(a+\mathfrak{p})/(b+\mathfrak{p}) \in \text{Frac}(A/\mathfrak{p})$ must be $(0+\mathfrak{p})/(b+\mathfrak{p}) = 0 \in \text{Frac}(A/\mathfrak{p})$. The image of the map is clearly all of $\kappa(\mathfrak{p})$ and so by the first isomorphism theorem we have $\text{Frac}(A/\mathfrak{p}) \cong \kappa(\mathfrak{p})$ as required. □

With reference to [1] and [4].

Q7. Reduced spectra quotient

Let A be a ring and let $J \subseteq A$ be an ideal such that every element is nilpotent, that is, for every $f \in J$ there is some $n > 0$ such that $f^n = 0$. We then define the canonical projection $\pi : A \to A/J$ such that $f \mapsto f + J$, which is clearly a ring homomorphism, and moreover is clearly surjective. We can then define the induced map $\pi^* : \text{Spec}(A/J) \to \text{Spec}(A)$ such that $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$. From the lemma in class, we know that due to the surjectivity of π, π^* is injective. Further, we know that

$$\pi^*(\text{Spec}(A/J)) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \ker(\pi)\} \,, \tag{1.7.1}$$

and since π is merely the quotient map we have $\ker(\pi) = J$. Further, we have a corollary from class stating: any $f \in A$ is nilpotent if and only if $f \in \mathfrak{p}$ for every prime ideal $\mathfrak{p} \subseteq A$. In other words, $J \subseteq \mathfrak{p}$ for all prime ideals $\mathfrak{p}$. Therefore,

$$\pi^*(\text{Spec}(A/J)) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq J\} = \text{Spec}(A) \,, \tag{1.7.2}$$

and so π^* is surjective and hence bijective as desired. □

2 Zariski topology

Q6. Spectra is quasi-compact

Let A be a ring where it's spectra is endowed with the Zariski topology with closed sets

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\} \tag{2.6.1}$$

for some subset $S \subseteq A$. We will prove that $\operatorname{Spec} A$ is a quasi-compact topological space; that is, every open cover of $\operatorname{Spec} A$ has a finite subcover. We know from lectures that the basic open sets

$$U_f = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\} = \operatorname{Spec} A \backslash V(f) \tag{2.6.2}$$

form a basis for the topology. Let $\{U_{f_i}\}_{i\in I}$ be an open cover of $\operatorname{Spec} A$, where $\operatorname{Spec} A = \cup_{i\in I} U_{f_i}$. Taking the complements of both sides with respect to $\operatorname{Spec} A$, this is equivalent to the condition

$$(\operatorname{Spec} A)^c = \varnothing = \left(\bigcup_{i\in I} U_{f_i}\right)^c = \bigcap_{i\in I} U_{f_i}^c = \bigcap_{i\in I} V(f_i)\,, \tag{2.6.3}$$

hence we want to show that there is a finite index set $J \subseteq I$ such that $\cap_{j\in J} V(f_j) = \varnothing$.

To make sense of this intersection, we note that for ideals I, J we have $V(I) \cap V(J) = V(I + J)$: if $\mathfrak{p} \in V(I) \cap V(J)$, i.e. $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \supseteq J$, then for any $i \in I$ and $j \in J$ we have $i + j \in \mathfrak{p}$ (since it is an ideal), so $I + J \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(I + J)$ so $V(I + J) \supseteq V(I) \cap V(J)$; for the reverse, suppose $\mathfrak{p} \in V(I + J)$ so $\mathfrak{p} \supseteq I + J$, then since $I \subseteq I + J$ and $J \subseteq I + J$ we see that $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \supseteq J$, i.e. $\mathfrak{p} \in V(I) \cap V(J)$, proving the claim. It is clear that this can be extended easily with induction, so

$$\bigcap_{i\in I} V(I_i) = V\Big(\sum_{i\in I} I_i\Big) \tag{2.6.4}$$

for ideals $I_i \subseteq A$. We also note from lectures that $V(f_i) = V((f_i))$, where (f_i) is the ideal generated by f_i. Putting this together, we deduce that our condition is the same as

$$\varnothing = \bigcap_{i\in I} V(f_i) = \bigcap_{i\in I} V((f_i)) = V\Big(\sum_{i=1}(f_i)\Big)\,. \tag{2.6.5}$$

Also from lectures, $V(S) = \varnothing$ if and only if $1 \in S$. By definition of the sum of ideals, any element of $\sum_{i\in I}(f_i)$ is a finite sum $g_{i_1} + \cdots + g_{i_r}$ for $g_{i_j} \in (f_i)$. Thus we have $1 = g_{i_1} + \cdots + g_{i_j}$ for some $g_{i_j} \in (f_i)$, meaning $\left(\sum_{j=1}^r g_j\right) = A$. Hence reversing all of the previous logic we have

$$\bigcup_{j=1}^r U_{f_j} = \operatorname{Spec} A \tag{2.6.6}$$

and we are done. □

With reference to [5].

Q10. Spectrum is sober

Let A be a ring. We will show that $\operatorname{Spec} A$ with the Zariski topology is a sober topological space, that is, every irreducible closed subset has a unique generic point. Taking our cues from the lecture notes, we will show that

$$\Phi : \operatorname{Spec} A \to \{\text{irreducible closed subsets of } \operatorname{Spec} A\} : \mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}} \tag{2.10.1}$$

is bijective. It is clear that the map is well defined since $\overline{\{\mathfrak{p}\}}$ is obviously closed by definition, and is irreducible because if $\overline{\{\mathfrak{p}\}} = Z_1 \cup Z_2$ for closed Z_1, Z_2, then $\mathfrak{p} \in Z_1$ or $\mathfrak{p} \in Z_2$, so $\overline{\{\mathfrak{p}\}} \subseteq Z_1$ or Z_2 by definition of the closure. If we can show that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, this will show that every irreducible closed subset of $\operatorname{Spec} A$ is of the form $\overline{\{\mathfrak{p}\}}$, hence showing that $\mathfrak{p}$ is a generic point of $\operatorname{Spec} A$.

To first show that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, i.e. showing that $V(\mathfrak{p})$ is irreducible, suppose that $V(\mathfrak{p}) = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed. Then by theorem 10.4 of the lecture notes, we know that there exist radical ideals $J_i \subseteq A$ such that $Z_1 = V(J_1)$ and $Z_2 = V(J_2)$. Define

$$I(W) = \bigcap_{\mathfrak{p} \in W} \mathfrak{p}\,, \tag{2.10.2}$$

then clearly we have $I(V(W)) = W$ as it is just the intersection of supersets of W. Noting that we also have $V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$ by basic properties of the Zariski sets, we see that

$$\mathfrak{p} = I(V(\mathfrak{p})) = I(V(J_1) \cup V(J_2)) = I(V(J_1 \cap J_2)) = J_1 \cap J_2\,. \tag{2.10.3}$$

So $\mathfrak{p} = J_1 \cap J_2$ and since by the definition of $J_1 J_2$ (the set of finite sums $i_1 j_1 + \cdots + i_k j_k$ where $i_h \in J_1$ and $j_h \in J_2$) we have $J_1 J_2 \subseteq J_1 \cap J_2$, we see that $\mathfrak{p} \supseteq J_1 J_2$ and since $\mathfrak{p}$ is prime we must have $\mathfrak{p} \supseteq J_1$ or J_2. Hence, $V(\mathfrak{p}) \subseteq Z_1$ or Z_2 and hence is irreducible, thus $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$.

To show the opposite direction, that is, that any closed irreducible set $Z \subseteq \operatorname{Spec} A$ is of the form $V(J)$ for some *prime* ideal J, suppose $Z = V(J)$ for some radical ideal $J \subseteq A$. Suppose $ab \in J$, then by theorem 10.1 of the lectures, we see that $V(ab) = V(a) \cup V(b)$ and by assumption we have $V(ab) \supseteq J$. Therefore since $V(J)$ is irreducible, we have $V(J) \subseteq V(a)$ or $V(b)$. Noting that $V(a) = V((a)) = V(\sqrt{(a)})$ and that $\sqrt{J} = J$ by assumption, this tells us that $\sqrt{(a)} \subseteq \sqrt{J}$ or $\sqrt{(b)} \subseteq \sqrt{J}$, so either $a \in J$ or $b \in J$ and so J is prime. Thus, $Z = V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ and so we are done. □

3 Modules

Q9. Annihilator properties

Let A be a ring. Let M be an A-module. For some $m \in M$, define the annihilator as

$$\mathrm{Ann}_A(m) = \{a \in A : am = 0\}. \tag{3.9.1}$$

Let $S \subseteq A$ be a multiplicative set.

Part a)

Noting that it is clear that $\mathrm{Ann}_A(m)$ is a subring, we can define

$$S^{-1}(\mathrm{Ann}_A(m)) = \left\{ \frac{a}{s} \;\middle|\; \begin{array}{l} a \in \mathrm{Ann}_A(m), s \in S \text{ and} \\ a_1/s_1 = a_2/s_2 \iff \exists t \in S \quad \text{s.t.} \quad t(s_2 a_1 - s_1 a_2) = 0 \end{array} \right\},$$

$$\mathrm{Ann}_{S^{-1}A}(m/1) = \left\{ a/s \in S^{-1}A : (a/s) \cdot (m/1) = 0 \right\}. \tag{3.9.2}$$

If $x \in S^{-1}\mathrm{Ann}_A(m)$, then $x = a/s \in S^{-1}A$ and in particular

$$\left(\frac{a}{s}\right) \cdot \left(\frac{m}{1}\right) = \left(\frac{1}{s}\right) \cdot \left(\frac{am}{1}\right) = \left(\frac{1}{s}\right) \cdot \left(\frac{0}{1}\right) = 0\,, \tag{3.9.3}$$

so $x \in \mathrm{Ann}_{S^{-1}A}(m/1)$. If $x = a/s \in \mathrm{Ann}_{S^{-1}A}(m/1)$, then we have $am/s = 0/1$, so there exists a $t \in S$ such that $t(am - 0s) = tam = 0$, so $ta \in \mathrm{Ann}_A(m)$. But then for any $q \in S$ we must have $ta/q \in S^{-1}(\mathrm{Ann}_A(m))$, so in particular letting $q = t \in S$ we have $a/1 \in S^{-1}(\mathrm{Ann}_A(m))$, so $a \in \mathrm{Ann}_A(m)$ and so $x \in \mathrm{Ann}_{S^{-1}A}(m/1)$, hence showing the two sets are equal.

Part b)

Now define

$$\mathrm{Ann}_A(M) = \bigcap_{m \in M} \mathrm{Ann}_A(m) = \left\{ a \in A : \text{for all } m \in M\,, am = 0 \right\}. \tag{3.9.4}$$

Suppose M is a finitely generated module, that is, there exist finite generators $m_1, \ldots, m_n$ such that for any $m \in M$ there exist $k_1, \ldots, k_n \in A$ such that $m = k_1 m_1 + \cdots + k_n m_n$. Note that $S^{-1}M$ admits a natural structure as an A-module. Again comparing the definitions of our two sets we have

$$S^{-1}\mathrm{Ann}_A(M) = \left\{ \frac{a}{s} : \text{for all } m \in M, am = 0, \text{ and } s \in S\,, \text{and usual equiv-relation} \right\},$$

$$\mathrm{Ann}_{S^{-1}A}(S^{-1}M) = \left\{ \frac{a}{s} \in S^{-1}A : \forall \frac{m}{q} \in S^{-1}M, \left(\frac{a}{s}\right) \cdot \left(\frac{m}{q}\right) = 0 \right\}. \tag{3.9.5}$$

Suppose $x \in S^{-1}\mathrm{Ann}_A(M)$, so $x = a/s$ with the necessary properties. Then for any $m/q \in S^{-1}M$ we have

$$\frac{a}{s} \cdot \frac{m}{q} = \frac{1}{sq} \cdot \frac{am}{1} = \frac{1}{sq} \cdot 0 = 0\,, \tag{3.9.6}$$

so $x \in \mathrm{Ann}_{S^{-1}A}(S^{-1}M)$. Now suppose $x \in \mathrm{Ann}_{S^{-1}A}(S^{-1}M)$, where $x = a/s$ as required. Then there exists a $t \in S$ such that

$$t(am - 0sq) = tam = 0\,, \tag{3.9.7}$$

By assumption, since the property holds for all $m/q \in S^{-1}M$, we can choose $m_i/1$ where each m_i are the finite generators, and see that we have some t_i such that $t_i a m_i = 0$. Building this up, we can take $b = at_1 \dots t_n$, and thus $b/s \in S^{-1}\mathrm{Ann}_A(M)$ and so we are done. □

Part c)

We will prove $M = 0$ if and only if $\mathrm{Ann}_A(M) = A$. Suppose $M = 0$, then

$$\mathrm{Ann}_A(0) = \big\{a \in A : a0 = 0\big\} = A \tag{3.9.8}$$

since we are in a ring so this condition is true for all $a \in A$. Suppose that $\mathrm{Ann}_A(M) = A$, then for every $m \in M$ we must have $am = 0$ for all values of $a \in A$. In particular we can take $a = 1$, so $m = 1m = 0$ for all m, thus $M = 0$ as required. □

Q11. Support of tensor product

Let $(A, \mathfrak{m})$ be a local be a ring, and let M and N be finitely generated A-modules. Recall the residue field $\kappa(\mathfrak{m}) = \kappa = M/\mathfrak{m}A = (A - \mathfrak{m})^{-1}M$. (Note that $M \otimes_A N \cong N \otimes_A M$, so different orderings throughout may be safely ignored). Suppose that $M \otimes_A N \otimes_A \kappa(\mathfrak{m}) = 0$, we will show that either $M = 0$ or $N = 0$. We first note that in defining $M_\kappa = \kappa \otimes_A M$, we can consider a bliniear homomorphism

$$\Psi : (A/\mathfrak{m}) \otimes_A M \to M/\mathfrak{m}M : (a, x) \mapsto ax \bmod \mathfrak{m}M \tag{3.11.1}$$

which induces a linear homomorphism as a kind of inverse for Ψ,

$$\Phi : M \to (A/\mathfrak{m}) \otimes_A M : m \mapsto \bar{1} \otimes_A x \tag{3.11.2}$$

where $\bar{1} = \pi(1)$ for the canonical projection into $A/\mathfrak{m}$. Using the fact that $\mathfrak{m}M \subseteq \ker \Phi$, we see that we have an isomorphism

$$(A/\mathfrak{m}) \otimes_A M \cong M/\mathfrak{m}M\,. \tag{3.11.3}$$

This allows us to deduce that

$$M_\kappa = \kappa \otimes_A M = (A/\mathfrak{m}A) \otimes_A M \cong M/(\mathfrak{m}A)M = M/\mathfrak{m}M\,. \tag{3.11.4}$$

So our condition now becomes

$$M \otimes_A N \otimes \kappa \cong (A - \mathfrak{m})^{-1}(M \otimes_A N) = 0\,. \tag{3.11.5}$$

Using question 12 of the assignment, we then see that we have

$$(A - \mathfrak{m})^{-1}(M \otimes_A N) \cong (A - \mathfrak{m})^{-1}M \otimes_A (A - \mathfrak{m})^{-1}N = M_\kappa \otimes_A N_\kappa = 0\,. \tag{3.11.6}$$

But since κ is a field, we see that both M_κ and N_κ are just vector spaces over a field. Using the simple property that $\dim(M_\kappa \otimes_A N_\kappa) = \dim(M_\kappa)\dim(N_\kappa)$, we see that our condition implies $\dim(M_\kappa) = 0$ or $\dim(N_\kappa) = 0$. Supposing without loss of generality that $M_\kappa = 0$, since $\mathfrak{m}$ is a maximal ideal this tells us that $M = \mathfrak{m}M$. Since $\mathfrak{m}$ is unique by construction of the local ring, it will be equal to the Jacobson radical of A. By the Nakayama lemma, this then implies that $M = 0$ as required. □

With reference to [3] and [6].

4 Algebras

Q4. Finite A-Algebras

Let A be a ring and B an A-algebra. We say that B is finite if B is finitely generated as an A-module. That is, there is a surjection $q : A^{\oplus r} \twoheadrightarrow B$ for some fixed $r \in \mathbb{N}_{>0}$

Part a)

Considering $B = A[x]$, it is clear that this is not a finite A-algebra. The canonical surjection would be: given $p(x) \in A[x]$ such that $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, we would clearly like to define $q(a_0, \ldots, a_n) = p(x)$, hence setting $r = n$. But then we can always find a higher degree polynomial $p'(x) = p(x) + x^{n+1}$, hence breaking the surjection and showing that there is no finite r to make it work. Thus $A[x]$ is not finite.

Part b)

By contrast, if $B = A[x]/(x^2)$ where

$$A[x]/(x^2) = \left\{p(x) \in A[x] : x^2 = 0\right\} = \left\{p(x) \in A[x] : p(x) = a + bx \ \text{ for } \ a, b \in A\right\}, \tag{4.4.1}$$

then this is a finite algebra as we have the obvious surjection

$$q : A \oplus A \twoheadrightarrow A[x]/(x^2) : (a, b) \mapsto a + bx\,. \tag{4.4.2}$$

Part c)

We first note the following useful lemma of Atiyah-Macdonal: "An A-algebra B is finite if and only if it is isomorphic to a quotient $A^{\oplus r}/M$ by an A-submodule $M \subset A$." We probably won't explicitly refer to it, but its nice to write down anyway.

Our first A-algebra, taking our cues from part b), is $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$. Clearly this is an A-algebra since $\mathbb{R}[x]$ is, and the surjection $q : \mathbb{R} \oplus \mathbb{R} \twoheadrightarrow \mathbb{R}[x]/(x^2 + 1)$ is the same as in (4.4.2).

Further, the matrices $\mathrm{Mat}_n(\mathbb{R})$ form an A-algebra with the standard addition and multiplication. Then define

$$q : \mathbb{R}^{\oplus n} \twoheadrightarrow \mathrm{Mat}_n(\mathbb{R}) : (a_1, \dots, a_{n^2}) \mapsto \begin{pmatrix} a_1 & \dots & a_n \\ \vdots & \ddots & \vdots \\ a_{n(n-1)+1} & \dots & a_{n^2} \end{pmatrix} \tag{4.4.3}$$

Again it is clear that this is a surjection.

Finally, using this matrix example, we can also consider the quarternions $\mathbb{H}$. Since they are generated by particular Pauli matrices in $\mathrm{Mat}_4(\mathbb{R})$, we again have our surjection

$$q : \mathbb{R}^{\oplus 4} \twoheadrightarrow \mathrm{Mat}_4(\mathbb{R}) : (a, b, c, d) \mapsto \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} . \tag{4.4.4}$$

Note: it didn't feel like cheating to include this example despite the matrices above because quarternions are genuinely interesting algebraic structures, its just that their matrix definition is the easiest to write down for me.

Part d)

Suppose B is a finite A-algebra with surjection $p : A^{\oplus r} \twoheadrightarrow B$, and $q : B \twoheadrightarrow C$ is a surjection of A-algebras. Then we have the following situation:

$$A^{\oplus r} \overset{p}{\twoheadrightarrow} B \overset{q}{\twoheadrightarrow} C \,, \qquad A^{\oplus r} \overset{f}{\dashrightarrow} C \tag{4.4.5}$$

Setting $f = q \circ p : A^{\oplus r} \twoheadrightarrow C$, we see that f is a well defined surjective map since for any $c \in C$ we have $b \in B$ with $q(b) = c$, and for any $b \in B$ we have $a \in A^{\oplus r}$ with $p(a) = b$, so $f(a) = q(p(a)) = c$. Therefore C is also a finite algebra.

Q6. Integral A-algebras

Let A be a ring and let B be an integral A-algebra, that is, for all $b \in B$ there exists a monic polynomial $p(x) \in A[x]$ such that $p(b) = 0$.

Part a)

Suppose C is an A-algebra, then we can form the C-algebra $B \otimes_A C$ as defined in class with the multiplication map

$$\cdot : (B \otimes_A C) \times (B \otimes_A C) \to B \otimes_A C : (b_1 \otimes c_1, b_2 \otimes c_2) \mapsto (b_1 b_2 \otimes c_1, c_2) \,. \tag{4.6.1}$$

In this way we can naturally take polynomials of tensor product elements. To show $B \otimes_A C$ is an integral C-algebra, suppose $b \otimes c \in B \otimes_A C$ is a *pure* tensor. Since B is integral, we have a monic $p(x) \in A[x]$ such that for our chosen b value we have

$$p(b) = b^n + a_{i-1} b^{n-1} + \dots + a_0 = 0 \,. \tag{4.6.2}$$

Applying this to $b \otimes c$ we see that

$$\begin{aligned} p(b \otimes c) &= (b \otimes c)^n + a_{n-1}(b \otimes c)^{n-1} + \cdots + a_0(1 \otimes 1) \\ &= (b^n \otimes c^n) + a_{n-1}(b^{n-1} \otimes c^{n-1}) + \cdots + a_0(1 \otimes 1) \,. \end{aligned}$$

This seems problematic, until we view $B \otimes_A C$ as a C-algebra where $c \mapsto 1 \otimes c$. This then allows us to effectively take coefficients of $k \in C$ where

$$k(b \otimes c) = k(b \otimes (1 \otimes c)) = (b \otimes k(1 \otimes c)) = (b \otimes (1 \otimes kc)) = (b \otimes kc) \,. \tag{4.6.3}$$

We see that in adjusting $p(x)$ slightly, we can let $q(x) \in C[x]$ to now have coefficients $a_i k_i$ where $k_i = c^{n-i}$ (where the product is well defined since C is an A-algebra). Then we have

$$\begin{aligned} q(b \otimes c) &= (b^n \oplus c^n) + c a_{n-1}(b^{n-1} \otimes c^{n-1}) + c^2 a_{n-2}(b^{n-2} \otimes c^{n-2}) + \cdots + c^n a_0(1 \otimes 1) \\ &= (b^n \oplus c^n) + a_{n-1}(b^{n-1} \otimes c^n) + a_{n-2}(b^{n-2} \otimes c^n) + \cdots + a_0(1 \otimes c^n) \\ &= (p(b) \otimes c^n) = (0 \otimes c^n) = 0(1 \otimes c) = 0 \,, \end{aligned} \tag{4.6.4}$$

thus showing that $q(x)$ is a monic polynomial satisfying $q(b \otimes c) = 0$ for all pure tensors $b \otimes c \in B \otimes_A C$. If we have a non-pure tensor in $B \otimes_A C$, then it will just be a sum of pure tensors. By Q5, we know that the integral elements form a subring. Since every pure tensor is integral, we now have that their sums are also integral, hence every element of the tensor product is integral, thus showing $B \otimes_A C$ is an integral C-algebra. □

Part b)

Suppose $S \subseteq A$ is a multiplicative set. To show $S^{-1}B$ is an integral $S^{-1}A$-algebra, let $b/s \in S^{-1}B$ where $p(x) \in A[x]$ is such that $p(b) = 0$. Proceeding in a very similar manner to the above, let $q(x) \in S^{-1}A$ now have coefficients $a_i k_i$ where $k_i = 1/s^{n-i} \in S^{-1}A$. Then

$$q(b/s) = \frac{b^n}{s^n} + \frac{a_{n-1}}{s}\frac{b^{n-1}}{s^{n-1}} + \cdots + \frac{a_0}{s^n} \tag{4.6.5}$$

$$= \frac{1}{s^n}(b^n + a_{n-1}b^{n-1} + \cdots + a_0) \tag{4.6.6}$$

$$= \frac{1}{s^n}(0) = 0 \,, \tag{4.6.7}$$

hence showing that $S^{-1}B$ is an integral $S^{-1}A$-algebra. □

5 Chain conditions

Q1. Noetherian modules

Let A be a ring. Consider an exact sequence of A-modules:

$$0 \longrightarrow M'' \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0 \ . \tag{5.1.1}$$

Part a)

We will prove that if M is noetherian, then so is M'' and M'. Suppose M is noetherian, that is, every ascending chain of A-submodules $M_1 \subseteq M_2 \subseteq \dots$ eventually stabilies, i.e. there is some $k \in \mathbb{N}$ such that for each $n \geq k$ we have $M_n = M_{n+1}$. Since we have our exact sequence, f is injective which means we can treat M'' as a submodule of M. But any submodule M'' of a noetherian module M is clearly noetherian since each ascending chain element is a subset of one in M and so also must stabilise, so M'' is noetherian.

Further, we know by exactness that $M/\mathrm{im}(f) = M/f(M'') \cong M'$, so if we can show any quotient of a noetherian module is noetherian then we will be finished. By the lattice isomorphism theorem, we know that A-submodules of $M/f(M'')$ are A-submodules N where $f(M'') \subseteq N \subseteq M$, hence this clearly preserves the ascending chain condition and so the quotient is noetherian, that is, M' is a noetherian module.

Part b)

Suppose now that M'' and M' are noetherian - we want to show that M is noetherian. Suppose $M_1 \subseteq M_2 \subseteq \dots$ is a chain of submodules $M_i \subseteq M$. Then since f and g are module homomorphisms, we can consider an ascending chain of submodules $f^{-1}(M_i) \subseteq M''$ and $g(M_i) \subseteq M'$,

$$f^{-1}(M_1) \subseteq f^{-1}(M_2) \subseteq \dots \qquad \text{and} \qquad g(M_1) \subseteq g(M_2) \subseteq \dots \ . \tag{5.1.2}$$

By assumption, we then have some $k_f, k_g \in \mathbb{N}$ for which these chains respectively stabilise. Letting $k = \max k_f, k_g$, for all $n \geq k$ we have $f^{-1}(M_n) = f^{-1}(M_{n+1})$ and $g(M_n) = g(M_{n+1})$. Then due to our assumed exactness in (5.1.1) we have two exact rows here since each element is a sub-module of M'', M and M' respectively given by the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(M_n) & \xrightarrow{f} & M_n & \xrightarrow{g} & g(M_n) & \longrightarrow & 0 \\ & & \| & & \downarrow \iota & & \| & & \\ 0 & \longrightarrow & f^{-1}(M_{n+1}) & \xrightarrow{f'} & M_{n+1} & \xrightarrow{g'} & g(M_{n+1}) & \longrightarrow & 0 \end{array} \ . \tag{5.1.3}$$

Since $M_n \subseteq M_{n+1}$, the map ι is just the inclusion. Then by the five lemma, which states that for a diagram of the above form due to the isomorphism (equality) between $f^{-1}(M_n)$ and $f^{-1}(M_{n+1})$ and respectively the other equality, then ι is an isomorphism. That is, $M_n = M_{n+1}$ for all $k \geq n$, so the ascending chain condition holds! That is, M is indeed noetherian. □

With reference to [8].

Q9. Associated primes of filtration

Let A be a ring and let M be an A-module.

Part a)

Suppose M admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M\,, \tag{5.9.1}$$

where M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \subseteq A$ prime. We will show that $\operatorname{Ass}_A(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ using induction on n, where

$$\operatorname{Ass}_A(M) = \big\{\mathfrak{p} \in \operatorname{Spec} A : \exists m \in M \text{ s.t. } \operatorname{Ann}_A(m) = \mathfrak{p}\big\}\,. \tag{5.9.2}$$

Our inductive statement $S(n)$ is: for all $j \leq n$, if $\mathfrak{p} = \operatorname{Ann}_A(m_j)$ for some $m_j \in M_j$, then $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. The argument for the base case $S(1)$ will be identical to that given below for $S(n-1)$.

Suppose $S(n-1)$ is true and let $\mathfrak{p} \in \operatorname{Ass}_A(M)$, where $m \in M = M_n$ is the element which it annihilates. If $m \in M_{n-1}$, then $[m] = 0 \in M_n/M_{n-1} \cong A/\mathfrak{p}_n$, hence via our isomorphism

$$\Phi : A \to M_n/M_{n-1} : a \mapsto am \tag{5.9.3}$$

we must have $\mathfrak{p}_n m = 0$, so $\mathfrak{p}_n \in \operatorname{Ass}_A(M)$. If $m \notin M_{n-1}$, then $[m] \neq 0 \in M_n/M_{n-1} \cong R/\mathfrak{p}_n$, but since $\mathfrak{p}$ is its annihilator we must have $\mathfrak{p} \subset \mathfrak{p}_n$. Since this is proper, we can find an $x \in \mathfrak{p}_n$ with $x \notin \mathfrak{p}$. Then $\mathfrak{p}$ is still an annihilator of $xm \in M$ since $\mathfrak{p}(xm) = x(\mathfrak{p}m) = 0$. By the assumed isomorphism $M_n/M_{n-1} \cong R/\mathfrak{p}_n$, we have $xm \in M_{n-1}$. By the inductive hypothesis, this means that $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}\}$. Therefore $S(n-1)$ implies $S(n)$ and so we are done. □

Part b)

Suppose A is noetherian and M is finitely generated. To show that $\operatorname{Ass}_A(M)$ is finite, we want to show that M satisfies the hypothesis of part a), that is, there is a finite filtration where each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \subseteq A$.

We know from Q8 of the worksheet that there are no associated primes if and only if $M = 0$, so supposing $M \neq 0$ we can find an associated prime $\mathfrak{p}_i = \operatorname{Ann}_A(m_1)$ for some nonzero m_1. Applying the first isomorphism theorem, we can take

$M_1 = Am_1 \cong A/\mathfrak{p}_1$. If $M_1 = M$ then we are done, so suppose $M \neq M_1$, meaning $M/M1$ is nonzero and so again by Q8 we must have some nonzero $m_2 \in M$ with $m_2 \notin M_1$ with associated prime $\mathfrak{p}_2 = \mathrm{Ann}_A(m_2 + M_1)$. If we then construct the map

$$\Psi : A \to M_2/M_1 : a \mapsto am_2 + M_1 \,, \tag{5.9.4}$$

then by the annihilation property above we must have $M_2/M_1 \cong A/\mathfrak{p}_2$ by the first isomorphism theorem. We can then continue this production inductively, and crucially we know that this process *will* terminate after finitely many steps since M is noetherian, therefore eventually $M_i/M_{i-1} = 0$ for all $i > n$ for some fixed $n \in \mathbb{N}$. Therefore, $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ and so by part a) we see that $\mathrm{Ass}_A(M)$ is finite. □

With reference to [7] and [2].

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