Functional Analysis Assignment 3

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Q1. Baire Category holds for open subset

It's easy to chase your tail when dealing with such proofs, so lets first get our definitions in order. For a topological space $(X, \mathcal{T})$, we define:

- $U \subseteq X$ is open if $U \in \mathcal{T}$. (1.1)
- $U \subseteq X$ is closed if U^c is open. (1.2)
- For $U \subseteq X$, the interior is $U^\circ = \bigcup\{V \subseteq U \mid V \text{ is open in } X\}$. (1.3)
- For $U \subseteq X$, the closure is $\overline{U} = \bigcap\{W \supseteq U \mid W \text{ is closed in } X\}$.
  or equivalently $\overline{U} = U \cup \{x = \lim_{n\to\infty} x_n \mid \forall n \in \mathbb{N},\ x_n \in U\}$ (1.4)
- For $U \subseteq X$, the boundary is $\partial U = \overline{U} \setminus U^\circ = \overline{U} \cap \overline{(X \setminus U)}$. (1.5)
- $U \subseteq X$ is dense in X if $\overline{U} = X$. (1.6)
- $U \subseteq X$ is nowhere dense in X if $(\overline{U})^\circ = \varnothing$. (1.7)
- $U \subset X$ is first category (also called meager) in X if U is a countable union of nowhere dense sets. U is second category if it is not first category. (1.8)
- $U \subset X$ is generic if U^c is first category. (1.9)

The Baire Cateogry Theorem states that a complete metric space X is *never* the countable union of nowhere dense sets, that is, X is always second category. Suppose that $X_0 \subset X$ is an open subset of X - we will show that the conclusion of Baire holds for this too.

We will first establish some necessary facts. Using the definition of the boundary $\partial X_0 = \overline{X_0} \setminus X_0^\circ$ we can re-express the closure in terms of something more friendly, namely we can calculate

$$\partial X_0 \cup X_0^\circ = (\overline{X_0} \setminus X_0^\circ) \cup X_0^\circ = (\overline{X_0} \cup X_0^\circ) \setminus (X_0^\circ \setminus X_0^\circ) = \overline{X_0}\,, \tag{1.10}$$

where the last equality was because $X_0^\circ \subset \overline{X_0}$ and clearly $X_0^\circ \setminus X_0^\circ = \varnothing$. We can then use the fact that since X_0 is open, $X_0^\circ = X_0$ and so we can write

$$\overline{X_0} = X_0 \cup \partial X_0\,. \tag{1.11}$$

The strategy from here is to show that ∂X_0 is a nowhere dense set. We first notice that ∂X_0 is a closed set since we can equivalently write $\partial X_0 = \overline{X_0} \cap \overline{(X \backslash X_0)}$, and since the closure of any set is closed, we have that the boundary is the intersection of two closed sets and so is itself closed. This means $(\overline{\partial X_0})^\circ = (\partial X_0)^\circ$, so now we just need to show that the boundary has empty interior.

Since X_0^c is closed, we know that its closure is equal to itself, hence from (1.5) we can equivalently write $\partial X_0 = \overline{X_0} \cap X_0^c$, and then since the interior of a finite intersection is the intersection of the interiors, we can write

$$\partial X_0^\circ = (\overline{X_0} \cap X_0^c)^\circ = (\overline{X_0})^\circ \cap (X_0^c)^\circ \,. \tag{1.12}$$

The last piece of the puzzle that we need is the fact that for any set $U \subset X$ we have $(U^c)^\circ = (\overline{U})^c$:

$$\overline{U}^c = \left(\bigcap \{W \supseteq U \mid W \text{ is closed in } X\}\right)^c = \bigcup \{W^c \subseteq U^c \mid W^c \text{ is open in } X\} = (U^c)^\circ \,. \tag{1.13}$$

Therefore, returning to (1.12) we see that

$$\partial X_0^\circ = (\overline{X_0})^\circ \cap (X_0^c)^\circ = (\overline{X_0})^\circ \cap (\overline{X_0})^c \subset \overline{X_0} \cap (\overline{X_0})^c = \varnothing \,. \tag{1.14}$$

This gives us that $(\overline{\partial X_0})^\circ = \partial X_0^\circ = \varnothing$ and so the boundary of an open set $X_0 \subset X$ is nowhere dense. We now have all the ingredients we need.

For a contradiction, suppose X_0 is the countable union of nowhere dense sets F_n, so

$$X_0 = \bigcup_{n=1}^{\infty} F_n \,. \tag{1.15}$$

From (1.11), we have that

$$\overline{X_0} = \left(\bigcup_{n=1}^{\infty} F_n\right) \cup \partial X_0 \,, \tag{1.16}$$

meaning $\overline{X_0}$ is a countable union of nowhere dense sets. But from the Baire Category Theorem, since $\overline{X_0}$ is closed and hence a complete metric space in its own right, it cannot be first category. Hence we have derived a contradiction and so the open $X_0 \subset X$ must also be second category, hence proving the statement. $\square$

Q2. All the categories

Suppose X is a complete metric space, $F \subset X$ is a closed subset and $O \subset X$ is open.

Part a)

Suppose F has an empty interior $F^\circ = \varnothing$. We know that for any topological space we have $\overline{F} = F$. Hence, $(\overline{F})^\circ = F^\circ = \varnothing$ and so F itself is nowhere dense, hence it is clearly a countable union of one nowhere dense set.

Suppose F is first category and can be written as $F = \bigcup_{n=1}^{\infty} F_n$ where each F_n is nowhere dense. Assume F has a non-empty interior for a contradiction. Then there is an open ball $B_\varepsilon(x) \subset F$ that is covered by the sets F_n, which means we can write $B_\varepsilon(x) = \bigcup_{n=1}^{\infty} F_n \cap B_\varepsilon(x)$. But we can clearly see that each $F_n \cap B_\varepsilon(x)$ is nowhere dense since we can write

$$(\overline{F_n \cap B_\varepsilon(x)})^\circ \subset (\overline{F_n} \cap \overline{B_\varepsilon(x)})^\circ = (\overline{F_n})^\circ \cap (\overline{B_\varepsilon(x)})^\circ = \varnothing \cap (\overline{B_\varepsilon(x)})^\circ = \varnothing\,, \tag{2.1}$$

meaning that we have now shown $B_\varepsilon(x)$ must be first category. But by Baire's theorem, and the first question in particular, $B_\varepsilon(x)$ cannot first category as it is an open subset of a complete metric space. Hence, we have a contradiction and so F must have an empty interior, so $F^\circ = \varnothing$. So a closed F is first category if and only if it has empty interior.

Part b)

Clearly, if $O = \varnothing$ then O is first category as the empty set is obviously nowhere dense. If O is first category, then since $O \subset X$ is an open subset of a complete metric space, we know from the first question that it is also second category. The only element in the intersection of the set of first category sets and the set of second category sets is $\varnothing$, so $O = \varnothing$. Therefore an open O is of first category if and only if it is empty.

Part c)

Suppose F is generic, then F^c is first category. By part b), since F^c is open it must be empty so $F^c = \varnothing$ so $F = X$. If $F = X$, then $F^c = X^c = \varnothing$ so F^c is first category meaning F is generic.

Suppose O is generic, so O^c is first category. O^c is a closed set so using part a) it must have an empty interior. If O^c contains no interior, then O is first category, meaning O is generic.

Q3. $L^p([0,1])$ is nowhere dense in $L^1([0,1])$

Consider the Banach space $L^p([0,1])$, which we have defined as the completion of $C^0([0,1])$ with respect to the p-norm $\|f\|_{L^p} = (\int_0^1 |f|^p)^{1/p}$.

Part a)

Suppose $f \in L^p([0,1])$ for some $p > 1$, meaning $\|f\|_p$ exists. Hölder's inequality tells us that for $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$. In taking $g(x) = 1$ we have

$$\|fg\|_1 = \|f\|_1 \leq \|f\|_p \left(\int_0^1 (1)^{\frac{p}{p-1}} dx \right)^{\frac{p}{p-1}} = \|f\|_p \,, \tag{3.1}$$

which means that $\|f\|_1$ also exists, hence $f \in L^1([0,1])$ (i.e. $L^p([0,1]) \subset L^1([0,1])$) as required.

Part b)

We will now consider the set

$$L^1([0,1]) \backslash L^p([0,1]) = \{ f \in L^1([0,1]) \mid f \notin L^p([0,1]) \} \tag{3.2}$$

and show that it is generic. To do this, we will consider the inclusion map

$$\begin{aligned} \iota : L^p([0,1]) &\hookrightarrow L^1([0,1]) \\ f &\mapsto f \end{aligned} \tag{3.3}$$

which is clearly linear and continuous between the two Banach spaces $L^p([0,1])$ and $L^1([0,1])$, and from part a) we know that this is well defined. It is clear that ι is not surjective: take the canonical example of $g(x) = x^{-1/p} \in L^1([0,1])$ since $\|g\|_1 = (\frac{p}{p-1})^{1/p}$, but is not convergent in $L^p([0,1])$ which shows non surjectivity. We will use this fact to show that the image $\iota(L^p([0,1])) \subset L^1([0,1])$ is a meager subset.

If we return to the proof of the Open Mapping Theorem, we can actually prove a slightly more general results without changing the proof. We hypothesised that T was a surjective continuous linear map between Banach spaces X and Y. However, the only reason we used the surjectivity of T was to show that the image of X, namely $T(X) = Y = \bigcup_{n=1}^{\infty} T(B_X(n))$ was not first category by Baire's theorem since Y was a complete metric space. So, if we change our hypothesis to "the image of $T(X)$ is not first category", we can then follow the remainder of the proof of the open mapping theorem to state the following result: *If $T : X \to Y$ is a continuous linear map between Banach spaces whose image is second category, then T is open.*

From this statement, we can use it to deduce that under the same conditions, if T is an open linear map between normed linear spaces, then it is surjective. Since X and Y are open in their respective topologies, we know that $T(X) \subset Y$ is an open linear subspace - indeed it is all of Y, which we can show by contradiction.

Suppose $Y \backslash T(X)$ is non-empty. Let $y_0 \notin T(X)$, then since $T(X)$ is a vector subspace, $y_0 \neq 0$. We can then find a sequence $\alpha_n \in \mathbb{C}$ such that $\alpha_n \to 0$. Then for all n, the sequence $y_n = a_n y_0 \notin T(X)$, again due to the closure of multiplication in the subspace. But with this construction, we have found a sequence y_n whose elements are not in $T(X)$ but converges to $0 \in T(X)$. Hence since $Y \backslash T(X)$ doesn't contain its limit points, it is not closed. But we already have that $T(X)$ must be open by hypothesis causing the desired contradiction, showing that $Y \backslash T(X)$ is empty and so $T(X) = Y$. So any open map with these conditions is surjective.

Summarising all of the above arguments, we now have:

If $T : X \to Y$ is a continuous linear map between Banach spaces whose image is second category, then T is open and moreover, T is surjective.

Taking the contrapositive of this statement, we see that if T is not surjective, then its image is not second category, i.e is first category. Putting this together with our observation that the inclusion obeys all of the necessary hypotheses, we can conclude that since ι is not surjective, then $\iota(L^p([0,1])) = L^p([0,1]) \subset L^1([0,1])$ is first category in L^1. Therefore the complement $L^1([0,1]) \backslash L^p([0,1])$ is generic as required.
$\square$

Q4. Weak closure of unit sphere is unit ball

Let $(X, \|.\|)$ be an infinite dimensional Banach space with the weak topology, that is, the weakest topology on X in which each functional $\ell \in X^*$ is continuous. Consider the unit sphere and unit ball in X,

$$S = \{x \in X \mid \|x\| = 1\}\,, \tag{4.1}$$

$$\text{and} \;\; B = \{x \in X \mid \|x\| \leq 1\}\,. \tag{4.2}$$

We will show that in the weak topology, $\overline{S} = B$. We first note that our basis (neighbourhood) elements in the weak topology on X are of the form

$$N_{\lambda_1,\ldots,\lambda_n;\varepsilon;x_0} = \{x \in X \;\big|\; \forall i = 1,\ldots,n\,,\; |\lambda_i(x - x_0)| < \varepsilon, \quad \text{where } \lambda_i \in X^*\,\}\,. \tag{4.3}$$

We will start by showing that $B \subset \overline{S}$. Without loss of generality, we will consider a neighbourhood of $x_0 = 0 \in B$, denoted $N_{\lambda_1,\ldots,\lambda_n;\varepsilon}$ and show that $0 \in \overline{S}$ in the weak topology (where it clearly is not an element of $\overline{S} = S$ in the strong (norm) topology) - that is, a weak open neighbourhood of 0 must intersect S.

Let $\mathcal{O}$ be a weak open set containing $0 \in B$, then we know that we can find an open neighbourhood of the form $M = N_{\lambda_1,\ldots,\lambda_n;\varepsilon}$ for some finite collection of λ_i such that $M \subset \mathcal{O}$. The idea here is to find a 'line' (what turns out to be a hyperplane) in some λ_k direction that intersects with S - in other words, $|\lambda_i(k)| < \varepsilon$ for all $i = 1,\ldots,n$ and some $\|k\| = 1$. Let us consider the map

$$\begin{aligned} \Phi : X &\to \mathbb{C}^n \\ x &\mapsto (\lambda_1(x),\ldots,\lambda_n(x)) \end{aligned} \tag{4.4}$$

Due to the linearity of each λ_i, and the fact that each λ_i is bounded, we see that Φ is a bounded linear map. We can consider the kernel of this map,

$$\ker\Phi = \{x \in X \;\big|\; \Phi(x) = 0\} = \{x \in X \;\big|\; \forall i = 1,\ldots,n\,,\; |\lambda_i(x)| = 0\} = \bigcap_{i=1}^{n} \ker\lambda_i\,. \tag{4.5}$$

Since Φ is a linear operator, we can use the rank nullity theorem to determine the dimensionality of this kernel - clearly, since λ_i is linear, we already know that $0 \in \ker\Phi$. Since $\dim(X) = \infty$ by hypothesis, we see that (since $\dim(\mathrm{im}\Phi) \leq n$),

$$\dim(\ker\Phi) = \dim(X) - \dim(\mathrm{im}\Phi) \geq \dim(X) - n = \infty\,, \tag{4.6}$$

meaning we *can* find another $y \in X\backslash\{0\}$ such that $\Phi(y) = 0$. We can then rescale our newfound y to put it onto the unit sphere: first start by observing that for $\alpha \in \mathbb{C}$ (or whatever other field we're working in) and for this same y, we have for any λ_i in our finite intersection in M

$$|\lambda(\alpha y)| = |\alpha||\lambda_i(y)| = 0\,, \tag{4.7}$$

meaning that $\alpha y \in M$. Hence, we can choose $\alpha = 1/\|y\|$, meaning $\|\alpha y\| = 1$, so we choose αy to be our desired point on the sphere. Hence, since $\alpha y \in M$, we see that our open neighbourhood of 0, M, intersects S, hence $B \subseteq \overline{S}$.

To show that $\overline{S} \subseteq B$, i.e. $B^c \subseteq \overline{S}^c = (S^c)^\circ$, we want to find an element $x \in B^c$ (so $\|x\| > 1$) such that there exists a weak open neighbourhood around it that doesn't intersect the sphere S.

Take $x_0 \in B^c$, so $\|x_0\| > 1$. From the second corollary of the Hahn-Banach theorem in class, we know that for any element $x_0 \in X$ there exists a linear functional $\lambda \in X^*$ such that $|\lambda(x_0)| = \|\lambda\|_{X^*}\|x_0\| > \|\lambda\|_{X^*}$. Hence, we will use this λ and set $\varepsilon = |\lambda(x_0)| - \|\lambda\|_{X^*}$ to construct our weak open neighbourhood. We can consider the neighbourhood

$$N(\lambda;\varepsilon) = \{x \in X \mid |\lambda(x)| < \varepsilon = |\lambda(x_0)| - \|\lambda\|_{X^*}\}\,. \tag{4.8}$$

We can then translate this open neighbourhood to create another open neighbourhood

$$\mathcal{O} = x_0 + N(\lambda;\varepsilon) = \{x_0 + y \in X \mid y \in N(\lambda;\varepsilon)\} \tag{4.9}$$

Clearly $x_0 \in \mathcal{O}$. To show that any $y' = x_0 + y \in \mathcal{O}$ has norm $\|y'\| > 1$ (so doesn't intersect the sphere), we first note that

$$|\lambda(x_0 + y)| \geq \big|\, |\lambda(x_0)| - |\lambda(y)| \,\big|\,, \tag{4.10}$$

from the reverse triangle inequality, and then since $y \in N(\lambda;\varepsilon)$ we have that

$$|\lambda(y)| < |\lambda(x_0)| - \|\lambda\|_{X^*}\,, \qquad \text{so} \quad |\lambda(x_0)| - |\lambda(y)| > \|\lambda\|_{X^*} > 0\,, \tag{4.11}$$

and so combining these two things we get that

$$|\lambda(x_0 + y)| > \|\lambda\|_{X^*} = \sup\{|\lambda(x)| \mid x \in X \text{ and } \|x\| \leq 1\}\,. \tag{4.12}$$

Due to the above equivalent version of the operator norm, this tells us that we must have $\|x_0 + y\| > 1$. Hence $\mathcal{O}$ is a weak open neighbourhood of x_0 that doesn't intersect the unit sphere. Hence $\overline{S} \subseteq B$, thus concluding the proof. $\square$