Lie Algebras Assignment 1

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Due Date: 22^{nd} March 2021

Lecture 2

Q2. Unitary operator is injective but not necessarily surjective

Let $\mathcal{H}$ be a separable Hilbert space. A linear transformation $f : \mathcal{H} \to \mathcal{H}$ is unitary if $\langle f(\psi), f(\phi)\rangle = \langle \psi, \phi\rangle$ for all $\psi, \phi \in \mathcal{H}$ with the inner product given on $\mathcal{H}$. We will demonstrate that a unitary transformation is injective but not necessarily surjective.

Let $f : \mathcal{H} \to \mathcal{H}$ be a unitary linear transformation and suppose $f(\psi) = f(\phi)$ for some $\psi, \phi \in \mathcal{H}$. Since $f(\psi) - f(\phi) = 0$, we have

$$0 = \langle f(\psi) - f(\phi), f(\psi) - f(\phi)\rangle = \langle f(\psi - \phi), f(\psi - \phi)\rangle = \langle \psi - \phi, \psi - \phi\rangle \,, \tag{2.1}$$

where the second equality is due to the linearity of f and the third is by its unitarity (unsure if this is a word but we'll roll with it). So by the definiteness of an inner product space we have that $\psi - \phi = 0$, hence f is indeed injective.

To show that a unitary operator f is not necessarily surjective, suppose we set $\mathcal{H} = \ell^2$, the space of square summable infinite sequences with complex entries (so $\mathcal{H}$ is infinite dimensional). Define the following "right shift" transformation

$$\begin{aligned} f : \ell^2 &\to \ell^2 \\ (x_1, x_2, x_3, \dots) = x &\mapsto x' = (0, x_1, x_2, x_3, \dots)\,. \end{aligned} \tag{2.2}$$

It is trivial that f is linear using standard notions of addition and scalar multiplication on ℓ^2. Taking the standard inner product on ℓ^2 we see that

$$\langle f(x), f(y)\rangle = \sum_{i=0}^{\infty} \overline{x'_i} y'_i = \overline{x'_0} y'_0 + \sum_{i=1}^{\infty} \overline{x_i} y_i = \sum_{i=1}^{\infty} \overline{x_i} y_i = \langle x, y\rangle \,, \tag{2.3}$$

so f is a unitary function. We note that it is trivial that f is well defined since it will not change the square summability of a vector in $\mathcal{H}$. However, f is not surjective: suppose we have $y \in \ell^2$ such that $y_1 = 1$ and $y_i = 0$ for all $i > 1$. If f was surjective then we would have some $x \in \ell^2$ such that

$$f(x) = (0, x_1, x_2, \dots) = (1, 0, 0, \dots) = y \,, \tag{2.4}$$

but this is clearly a contradiction as the vectors disagree on the first entry. Therefore a unitary transformation is not necessarily surjective. □

Q4. U lifts nicely to U^{ext}

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B} = \{\zeta_k\}_{k=1}^{\infty}$ be a countable orthonormal dense basis on $\mathcal{H}$. Define $W \subset \mathcal{H}$ as

$$W := \{\psi \in \mathcal{H} \mid \langle \zeta_1, \psi \rangle \neq 0\}\,, \tag{4.1}$$

which we note is open, and by Exercise L2-9 is also dense in $\mathcal{H}$. Suppose we have a function $U : W \to \mathcal{H}$ that is either linear and unitary or antilinear and antiunitary. We proved in the lectures that such a function is uniformly continuous, hence due to the density of W in $\mathcal{H}$ we can invoke the universal property of complete metric spaces to lift U on to all of $\mathcal{H}$, that is, produce the following commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \overset{U^{\text{ext}}}{\dashrightarrow} & \mathcal{H} \\ {\scriptstyle\iota}\uparrow & \nearrow_{U} & \\ W & & \end{array}\,, \tag{4.2}$$

where U^{ext} is unique and uniformly continuous and is constructed, in a well defined manner (due to the MHS lemma), as follows. Given a vector $\psi \in \mathcal{H}$, and choosing a Cauchy sequence $\{\psi^{(n)}\}_{n=0}^{\infty} \subseteq W$ such that $W \ni \psi^{(n)} \to \psi \in \mathcal{H}$, we naturally define

$$U^{\text{ext}}(\psi) := \lim_{n\to\infty} U(\psi^{(n)})\,. \tag{4.3}$$

To show that U^{ext} is either linear and unitary or antilinear and antiunitary we divide into the two cases of U.

First suppose that U is linear and unitary. Suppose we have sequences $\psi^{(n)} \to \psi$, $\phi^{(n)} \to \phi$ where each $\psi^{(n)}, \phi^{(n)} \in W$ and $\psi, \phi \in \mathcal{H}$. Given some $\lambda, \mu \in \mathbb{C}$ we may define $(\lambda\psi + \mu\phi)^{(n)} := \lambda\psi^{(n)} + \mu\phi^{(n)}$. However, it is not guaranteed that this is in W, so we have some work to do.

We need to justify the fact that we can always find such sequences $\psi^{(n)}$ and $\phi^{(n)}$ for which their linear combination is always in W. Given we have already found such sequences that converge to ψ and $\phi \in \mathcal{H}$ respectively, we just need to alter our sequence in the case that $\lambda\psi^{(n)} + \mu\phi^{(n)} \notin W$ for some subset of indices $M \subseteq \mathbb{N}$ (with equality a genuine possibility). We may alter this subsequence defined by M by defining for each n

$$\hat{\psi}^{(n)} := \begin{cases} e^{\frac{1}{n}}\psi^{(n)} & \text{if } n \in M \\ \psi^{(n)} & \text{otherwise} \end{cases} \tag{4.4}$$

We see that we still have $\hat{\psi}^{(n)} \to \psi$ by standard limit laws. Further, given that $\psi^{(n)}$ and $\phi^{(n)} \in W$ must be nonzero, for those problematic $n \in M$ we have

$$\xi^{(n)} = \lambda\hat{\psi}^{(n)} + \mu\phi^{(n)} = \lambda\psi^{(n)} + \mu\phi^{(n)} + \lambda(e^{\frac{1}{n}} - 1)\psi^{(n)}\,,$$
$$\text{so } \langle \zeta_1, \xi^{(n)} \rangle = \langle \zeta_1, \lambda(e^{\frac{1}{n}} - 1)\psi^{(n)} \rangle \neq 0\,.$$

Therefore we may instead define our Cauchy sequence converging to ψ as our new $\hat{\psi}^{(n)}$. So without loss of generality we may assume that $\lambda\psi^{(n)} + \mu\phi^{(n)} \in W$ for all n.

With that technicality out of the way, basic operations of vectors and limit laws give us $\lambda\phi^{(n)} + \mu\psi^{(n)} \to \lambda\phi + \mu\psi$. Thus we have

$$\begin{aligned} U^{\text{ext}}(\lambda\psi + \mu\phi) = \lim_{n\to\infty} U((\lambda\psi + \mu\phi)^{(n)}) &= \lim_{n\to\infty} U(\lambda\psi^{(n)} + \mu\phi^{(n)}) \\ &= \lim_{n\to\infty} \Big(\lambda U(\psi^{(n)}) + \mu U(\phi^{(n)})\Big) \\ &= \lambda \lim_{n\to\infty} U(\psi^{(n)}) + \mu \lim_{n\to\infty} U(\phi^{(n)}) \\ &= \lambda U^{\text{ext}}(\psi) + \mu U^{\text{ext}}(\phi)\,, \end{aligned} \tag{4.5}$$

and so U^{ext} is linear. For unitarity, we may use the continuity of the inner product in each argument to see that

$$\begin{aligned} \langle U^{\text{ext}}(\psi), U^{\text{ext}}(\phi)\rangle &= \big\langle \lim_{n\to\infty} U(\psi^{(n)}), \lim_{m\to\infty} U(\phi^{(m)})\big\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle U(\psi^{(n)}), U(\phi^{(m)})\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle \psi^{(n)}, \phi^{(m)}\rangle \\ &= \big\langle \lim_{n\to\infty} \psi^{(n)}, \lim_{m\to\infty} \phi^{(m)}\big\rangle \\ &= \langle \psi, \phi\rangle\,, \end{aligned} \tag{4.6}$$

and so U^{ext} is unitary.

In the case that U is antilinear and antiunitary, we see that we need only very minor modifications. The third equality of (4.5) now becomes

$$\lim_{n\to\infty} U(\lambda\psi^{(n)} + \mu\phi^{(n)}) = \lim_{n\to\infty} \Big(\overline{\lambda} U(\psi^{(n)}) + \overline{\mu} U(\phi^{(n)})\Big)$$

and so following this through we get $U^{\text{ext}}(\lambda\psi+\mu\phi) = \overline{\lambda}U^{\text{ext}}(\psi)+\overline{\mu}U^{\text{ext}}(\phi)$ for antilinearity. Similarly, the third equality of (4.6) becomes

$$\lim_{n\to\infty}\lim_{m\to\infty} \langle U(\psi^{(n)}), U(\phi^{(m)})\rangle = \lim_{n\to\infty}\lim_{m\to\infty} \overline{\langle \psi^{(n)}, \phi^{(m)}\rangle}$$

and so following through, noting that complex conjugation is also continuous, we have $\langle U^{\text{ext}}(\psi), U^{\text{ext}}(\phi)\rangle = \overline{\langle\psi,\phi\rangle}$, hence antiunitarity.

To show that U^{ext} is bijective, we first use our data from Ex L2-2 to know that U is automatically injective. Suppose $U^{\text{ext}}(\psi) = U^{\text{ext}}(\phi)$ for $\psi, \phi \in \mathcal{H}$ as limits as before. Then

$$0 = U^{\text{ext}}(\psi - \phi) = \lim_{n\to\infty} U(\psi^{(n)} - \phi^{(n)}) \quad \text{so} \quad \lim_{n\to\infty} (\psi^{(n)} - \phi^{(n)}) = 0 \tag{4.7}$$

by the continuity of U. Hence taking the limit we have $\psi = \phi$, hence U^{ext} is injective.

For surjectivity, we first make precise how U acts on elements of our orthonormal dense basis $\mathcal{B} = \{\zeta_k\}_{k=1}^\infty$ (which due to linearity thus defines how it acts on any element of $\mathcal{H}$). From lectures, it was shown that under the appropriate hypotheses of Wigner's theorem, we must have $U(\zeta_k) = \eta_k\zeta_k'$ for some $\eta_k \in \mathcal{U}(1) = \{\eta \in \mathbb{C} : |\eta| = 1\}$ and another perfectly good orthonormal dense basis $\mathcal{B}' = \{\zeta_k'\}_{k=1}^\infty$. We claim that $\mathcal{B}_U = \{U(\zeta_k)\}$ is also an orthonormal dense basis. To see that it is orthonormal, note that

$$\langle U(\zeta_k), U(\zeta_l)\rangle = \langle \eta_k\zeta_k', \eta_l\zeta_l'\rangle = \overline{\eta_k}\eta_l\langle\zeta_k', \zeta_l'\rangle = |\eta_k|^2\delta_{k,l} = \delta_{k,l}\,. \tag{4.8}$$

Then, we are practically given density on a silver platter. Using MHS Theorem L21-10, suppose we have $\psi = \sum_{k=1}^{\infty} \beta_k \zeta'_k \in \mathcal{H}$ for some $\beta_k \in \mathbb{C}$ such that $\langle \psi, U(\zeta_k) \rangle = 0$ for all $k \in \mathbb{N}$. Then

$$0 = \langle \psi, U(\zeta_k) \rangle = \left\langle \sum_{l=1}^{\infty} \beta_k \zeta'_k, \eta_k \zeta'_k \right\rangle = \sum_{l=1}^{\infty} \overline{\beta_l} \eta_k \langle \zeta'_l, \zeta'_k \rangle = \overline{\beta_l} \eta_k \delta_{k,l} = \overline{\beta_k} \eta_k \,. \tag{4.9}$$

We can safely assume that all $\eta_k \neq 0$, hence we must have $\overline{\beta_k} = 0$ for all $k \in \mathbb{N}$ and so $\psi = 0$. Thus by the theorem we conclude that $\{U(\zeta_k)\}_{k=1}^{\infty}$ is a dense basis for $\mathcal{H}$.

Given some $\psi \in \mathcal{H}$ with $\psi^{(n)} \to \psi$, for each $\psi^{(n)}$ we can hence write (assuming U is linear, with the antilinear case being an obvious modification)

$$\psi^{(n)} = \sum_{k=1}^{\infty} \beta_k^{(n)} U(\zeta_k^{(n)}) = U\left(\sum_{k=1}^{\infty} \beta_k^{(n)} \zeta_k^{(n)}\right) \quad \text{for some } \beta_k^{(n)} \in \mathbb{C}\,, \tag{4.10}$$

$$\text{so set } \phi^{(n)} = \sum_{k=1}^{\infty} \beta_k^{(n)} \zeta_k^{(n)} \,.$$

Then since $\mathcal{B}$ is also a dense basis, we know that $\lim_{n\to\infty} \phi^{(n)} = \phi \in \mathcal{H}$ exists. Therefore we may take ϕ as our element of the domain to show that

$$U^{\text{ext}}(\phi) = \lim_{n\to\infty} U(\phi^{(n)}) = \lim_{n\to\infty} \psi^{(n)} = \psi \tag{4.11}$$

and so U^{ext} is surjective, hence bijective and we are done. $\square$