# Partial Differential Equations Assignment 4

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## L19 Q2 - Exponential is not tempered

Consider the distribution $F$ defined by $f(x) = e^x$ on $\mathbb{R}$,

$$F(\varphi) = \int_{\mathbb{R}} e^x \varphi(x) \, \mathrm{d}x \, .$$

We will show that $F$ is not a tempered distribution. Recall the criterion from L19 Q1 for $d = 1$, which says that $F$ is tempered if and only if there is an integer $N \in \mathbb{N}$ and a constant $A \in \mathbb{R}$ such that for all $R \geq 1$ and all test functions $\varphi \in C_0^\infty(\mathbb{R})$ supported in $x \in [-R, R]$

$$|F(\varphi)| \leq AR^N \sup_{\substack{x \in [-R,R] \\ 0 \leq \alpha \leq N}} \left| \partial_x^\alpha \varphi(x) \right| \, . \tag{1}$$

For a counterexample, let $\varphi(x)$ be the "nearly identity" bump function defined in Lecture 5 Q2c). That is, for any $R > \delta > 0$ there is a Schwarz function $\varphi(x)$ that is 1 on $[R+\delta, R-\delta]$, vanishes outside of $[-R, R]$, and is strictly monotone on $[-R, -R+\delta]$ and $[R-\delta, R]$. Such a function is seen in Fig. 1.

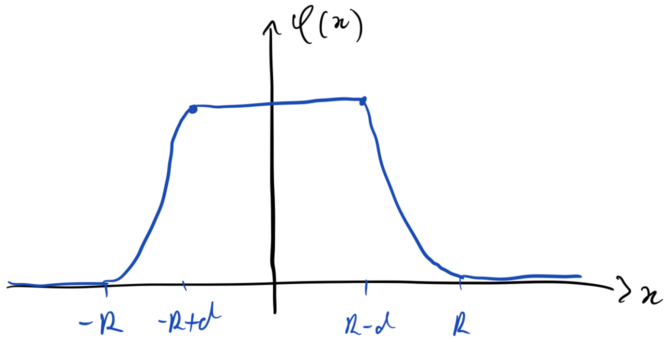


Figure 1: "Nearly identity" bump function $\varphi(x) \in C_0^\infty(\mathbb{R})$ supported on $[-R, R]$.

From that question we may take it as a given that this function is Schwarz, meaning that there is some constant $C > 0$ such that

$$2\sinh(R-\delta) = \sup_{\substack{x \in \mathbb{R} \\ 0 \leq \alpha, \beta < \infty}} \left| x^\beta \partial_x^\alpha \varphi(x) \right| < C \, .$$

In particular, this bound holds *independent of* $R$, which is the crucial point in applying this test function to show the polynomial growth fails. That is, for any choice of $R$ we have

$$\sup_{\substack{x\in[-R,R]\\0\le\alpha<\infty}} \left|\partial_x^\alpha \varphi(x)\right| < C\,.$$

We can find a lower bound for the left hand side of (1),

$$\begin{aligned}
|F(\varphi)| &= \left|\int_{-R}^{R} e^x \varphi(x)\,\mathrm{d}x\right| \\
&= \left|\int_{-R}^{-R+\delta} e^x \varphi(x)\,\mathrm{d}x + \int_{-R+\delta}^{R-\delta} e^x\,\mathrm{d}x + \int_{R-\delta}^{R} e^x \varphi(x)\,\mathrm{d}x\right| \\
&= \left|e^{R-\delta} - e^{-R+\delta} + \int_{R-\delta}^{R} (e^x + e^{-x})\varphi(x)\,\mathrm{d}x\right| \\
&\ge e^{R-\delta} - e^{-R+\delta}\,.
\end{aligned}$$

So the criteria (1) is satisfied if there exists a constant $A$ and integer $N$ such that

$$e^{R-\delta} - e^{-R+\delta} \le ACR^N\,.$$

But of course, exponentials grow faster than polynomials! To see this we may consider the limit of the quotient for any fixed even $N \in \mathbb{N}$. Then by repeated application of L'Hopital's rule,

$$\begin{aligned}
\lim_{R\to\infty} \frac{AR^N}{2\sinh(R-\delta)} &= \lim_{R\to\infty} \frac{ANR^{N-1}}{2\cosh(R-\delta)} \\
&= \lim_{R\to\infty} \frac{AN(N-1)R^{N-2}}{2\sinh(R-\delta)} \\
&= \dots \\
&= \lim_{R\to\infty} \frac{AN!}{2\sinh(R-\delta)} = 0\,.
\end{aligned}$$

Similarly for odd $N$ we have

$$\lim_{R\to\infty} \frac{AR^N}{2\sinh(R-\delta)} = \cdots = \lim_{R\to\infty} \frac{AN!}{2\cosh(R-\delta)} = 0\,.$$

Thus, $2\sinh(R-\delta)$ is not polynomially bounded and so for any constant $A \in \mathbb{R}$ and integer $N \in \mathbb{N}$ there will always exist an $R \in \mathbb{R}^+$ such that $e^{R-\delta} - e^{-R+\delta} > ACR^N$. Thus the criteria in (1) fails for all $N$ showing that $e^x$ is not a tempered distribution. $\square$