Random Matrix Theory Assignment 1

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Q1. Quaternion Matrices

Part a)

We will assume all of the data that has been given to us in the question. Let the Hermitian part of the self-dual matrix $Q \in \mathbb{C}^{2N\times 2N}$ be denoted as $H = Q + Q^\dagger$ where we know that $Q = \hat{\tau}_2 Q^T \hat{\tau}_2$. Also assume a quaternion vector $q \in \mathbb{H}^N$ obeys the symmetry condition $q^* = \hat{\tau}_2 q \tau_2$. Then

$$\hat{\tau}_2 H \hat{\tau}_2 = \hat{\tau}_2(Q + Q^\dagger)\hat{\tau}_2 = \hat{\tau}_2 Q \hat{\tau}_2 + \hat{\tau}_2 Q^\dagger \hat{\tau}_2 = Q^T + Q^* = (Q + Q^\dagger)^* = H^*$$

Hence showing that $H^* = \hat{\tau}_2 H \hat{\tau}_2$ as required.

Suppose that $v \in \mathbb{H}^N$ is one of the $2N$ eigenvectors of H with eigenvalue $\lambda \in \mathbb{C}$. Then $Hv = \lambda v$. We wish to use our conjugation condition to show that there is another eigenvector $v' \in \mathbb{H}^N$ that is orthogonal to v yet shares the same eigenvalue, hence showing that it is doubly degenerate. We notice that

$$\begin{aligned}
Hv &= \lambda v \\
\implies (Hv)^* &= (\lambda v)^* \\
\implies H^* v^* &= \lambda v^* \\
\implies \hat{\tau}_2 H \hat{\tau}_2 v^* &= \lambda v^* \\
\implies \hat{\tau}_2(\hat{\tau}_2 H \hat{\tau}_2 v^*) &= \hat{\tau}_2(\lambda v^*) \\
\implies H(\hat{\tau}_2 v^*) &= \lambda(\hat{\tau}_2 v^*)
\end{aligned}$$

Where we have used the fact that all eigenvalues λ of H must be real given that H is Hermitian. Also, $(\hat{\tau}_2)^2 = \mathbb{1}_{2N}$. Hence, we see that the vector $v' = \hat{\tau}_2 v \in \mathbb{H}^N$ is another vector that shares the the same eigenvalue λ. To prove orthogonality, we use the standard Frobenius inner product definition of an inner product on matrices.

Let $w_1, w_2 \in \mathbb{C}^N$ be arbitrary.

$$\begin{aligned}
\langle v, \hat{\tau}_2 v\rangle = \langle v, v\tau_2\rangle &= \operatorname{tr} v^\dagger v \tau_2 \\
&= \operatorname{tr} \begin{pmatrix} w_1^\dagger & -w_2^T \\ w_2^\dagger & w_1^T \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ -w_2^* & w_1^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \operatorname{tr} \begin{pmatrix} w_1^\dagger & -w_2^T \\ w_2^\dagger & w_1^T \end{pmatrix} \begin{pmatrix} iw_2 & -iw_1 \\ iw_1^* & iw_2^* \end{pmatrix} \\
&= iw_1^\dagger w_2 - iw_2^T w_1^* - iw_2^\dagger w_1 + iw_1^T w_2^* \\
&= iw_1^\dagger w_2 - (iw_1^\dagger w_2)^T + (iw_2^\dagger w_1)^T - iw_2^\dagger w_1 \\
&= iw_1^\dagger w_2 - iw_1^\dagger w_2 + iw_2^\dagger w_1 - iw_2^\dagger w_1 \\
&= 0
\end{aligned}$$

Hence, we can conclude that $\hat{\tau}_2 v$ is orthogonal to the original eigenvector v. Since $\hat{\tau}_2 v$ and v share the same eigenvalue λ and are orthogonal, and v was an arbitrary eigenvector, this proves that all eigenvalues of H are doubly degenerate. □

Part b)

We wish to calculate the quaternion Gaussian vector integral over the self-dual Q

$$\mathcal{G}(Q) = \int_{\mathbb{H}^N} d[q] \exp[-\operatorname{tr}(q-\eta)^T \hat{\tau}_2 Q (q-\eta)\tau_2]$$

by attempting to reduce it into the form of a Gaussian integral over a complex vector. First, we make the substitution $q \mapsto q - \eta \in \mathbb{H}^N$, where $d[q-\eta] = d[q]$ and the bounds of integration remain the same (i.e. over all of $\mathbb{H}^N$). Then using the properties as outlined in the question, we can calculate

$$\begin{aligned}
\operatorname{tr} q^T \hat{\tau}_2 Q q \tau_2 &= \operatorname{tr} q^T \hat{\tau}_2 Q (\hat{\tau}_2 q^*) \\
&= \operatorname{tr} q^T (Q^T) q^* \\
&= \operatorname{tr}(q^\dagger Q q)^T \\
&= \operatorname{tr} q^\dagger Q q
\end{aligned}$$

We can then calculate this quantity for an arbitrary $q \in \mathbb{H}^N$ (with $w_1, w_2 \in \mathbb{C}^N$), self-dual $Q \in \mathbb{C}^{2N\times 2N}$ where $X_1 \in \mathbb{C}^{N\times N}$ and $X_2, X_3 \in \operatorname{ASym}_{\mathbb{C}}(N)$.

$$\begin{aligned}
\operatorname{tr} q^\dagger Q q &= \operatorname{tr} \begin{pmatrix} w_1^\dagger & -w_2^T \\ w_2^\dagger & w_1^T \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_1^T \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ -w_2^* & w_1^* \end{pmatrix} \\
&= \operatorname{tr} \begin{pmatrix} w_1^\dagger & -w_2^T \\ w_2^\dagger & w_1^T \end{pmatrix} \begin{pmatrix} X_1 w_1 - X_2 w_2^* & X_1 w_2 + X_2 w_1^* \\ X_3 w_1 - X_1^T w_2^* & X_3 w_2 + X_1^T w_1^* \end{pmatrix} \\
&= w_1^\dagger (X_1 w_1 - X_2 w_2^*) - w_2^T (X_3 w_1 - X_1^T w_2^*) + w_2^\dagger (X_1 w_2 + X_2 w_1^*) + w_1^T (X_3 w_2 + X_1^T w_1^*) \\
&= (w_1^\dagger X_1 w_1 + w_1^T X_1^T w_1^*) + (w_2^\dagger X_1 w_2 + w_2^T X_1^T w_2^*) \\
&\quad + (w_2^\dagger X_2 w_1^* - w_1^\dagger X_2 w_2^*) + (w_1^T X_3 w_2 - w_2^T X_3 w_1) \\
&= 2w_1^\dagger X_1 w_1 + 2w_2^\dagger X_1 w_2 + 2w_2^\dagger X_2 w_1^* - 2w_2^T X_3 w_1
\end{aligned}$$

Here we have used the fact that $w_1^T X_1^T w_1^* = (w_1^\dagger X_1 w_1)^T = w_1^\dagger X_1 w_1$ since $w_1^\dagger X_1 w_1$ is a scalar. Also, $-w_1^\dagger X_2 w_2^* = w_1^\dagger X_2^T w_2^* = (w_2^\dagger X_2 w_1^*)^T = w_2^\dagger X_2 w_1^*$.

Unfortunately, despite staring at this for many many weeks, I have been unable to come up with a method of simplifying this expression into something more usable. Whilst the likes of the $w_1^\dagger X_1 w_1$ term look good, it appears to me that there is a factorisation method using the symmetry of Q where we would "wedge in" particular factors, e.g. something like $(w_1 - w_2)^\dagger (2X_1 + X_2 + X_3)(w_1 - w_2)$ (this one obviously doesn't work, its just an example of what I had been thinking). I also attempted to use the fact that $\operatorname{tr} q^\dagger Q q = \operatorname{tr} q^T Q^T q$ and exploit symmetry there - that, understandably, produced the same result as above. I then thought about decompositions of the self-dual Q but my research, despite providing interesting facts about the Schur decomposition of a the Hermitian self-dual H, provided nothing on Q. I then also tried to exploit the fact that $Q = \frac{1}{2}[(Q + Q^T) + (Q - Q^T)]$ but again, no luck. If X_2 and X_3 were symmetric as opposed to anti symmetric, then we would have been able to cancel these terms in our above calculation which would have left the terms $2w_1^\dagger X_1 w_1 + 2w_2^\dagger X_1 w_2$ which would result in a factorisable integral over w_1 and w_2. Of course, this doesn't account for the contributions of X_2 and X_3 so it cannot be true. Alas...

I then had the thought of potentially abusing some notation using a similar process to the complex case in the notes. For a quaternion vector $q \in \mathbb{H}^n$, we can consider it as be comprised of $q = C + jK$ for $C, K \in \mathbb{C}^{2N}$ - then we can define (which I have never seen written anywhere but I'm just gonna go with it) $\operatorname{Co}(q) = C$ and $\operatorname{Qu}(q) = K$ being the "Complex" and "Quaternion" parts of q. We can then define, in a slightly reversed way to the notes,

$$q = \begin{pmatrix} \operatorname{Co}(q) \\ \operatorname{Qu}(q) \end{pmatrix} = \begin{pmatrix} C \\ K \end{pmatrix} \qquad A = \begin{pmatrix} Q + Q^\dagger & j(Q - Q^\dagger) \\ -j(Q - Q^\dagger) & Q + Q^\dagger \end{pmatrix}$$

Under these circumstances, we would have

$$\det(A) = 2^{4N} \det(Q) = 2^{4N} \det(X_1) \det((X_1^{-1})^T - X_3 X_1^{-1} X_2)$$

Maybe, with any luck and very little conviction, some sort of formulation along these lines could lead to a result that looks kind of like

$$\mathcal{G}(Q) = \frac{(2\pi)^{2N}}{\sqrt{\det(A)}}$$

Unfortunately, this is the best I could muster up. I *cannot wait (pleasepleaseplease-pleaseplease)* to see the solution to this because it has puzzled me for weeks!

Part c)

Unfortunately due to my lack of concrete answer for part b), I also don't have a great deal of certainty as to why the result doesn't change for an arbitrary choice of $\eta \in \mathbb{C}^{2N \times 2}$. My best guess is that it has something to do with the analytic continuation arguments presented in the notes - however, this was dealing with a symmetric matrix C, so I am unsure why we are able to use this in our case. Nonetheless, this is a best guess.

Q2. Correlated Real Cauchy-Lorentz Ensemble

Consider the **correlated real Cauchy-Lorentz ensemble** which is given by the distribution

$$P(X|C) = \underbrace{\left(\frac{\det C^{p/2}}{\prod_{j=1}^{p} \pi^{n/2}\Gamma[\mu + j/2]/(2\Gamma[\mu + (j+n)/2])}\right)}_{A} \underbrace{\frac{1}{\det(\mathbb{1}_p + X^T C X)^{(p+n)/2+\mu}}}_{f(X)} \tag{2.1}$$

Where $\mu \geq 0$, Γ is the Gamma function, and $C \in \mathrm{Sym}_+(n)$.

Part a)

To check that $P(X|C)$ is normalised, we wish to show that $\int_{\mathbb{R}^{n\times p}} P(X|C) d[X] = 1$.

We first start by rescaling P. Since $C \in \mathrm{Sym}_+(n)$, we can write $C = D^T D$ for some $D \in \mathrm{GL}_{\mathbb{R}}(n)$. This also gives us $\det(C) = \det(D^T D) = \det(D)^2$. We can now make the substitution $U = DX$, where, for column vectors X_i we have

$$\begin{aligned} d[U] = d[DX] &= d[DX_1] \dots d[DX_p] \\ &= \det(D) d[X_1] \dots \det(D) d[X_p] \\ &= (\det(D))^p d[X_1] \dots d[X_p] \\ &= \det(C)^{p/2} d[X] \end{aligned}$$

and the support on U remains as $\mathbb{R}^{n\times p}$. Then

$$\begin{aligned} \int_{\mathbb{R}^{n\times p}} P(X|C) d[X] &= A \int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + X^T C X)^{(p+n)/2+\mu}} \\ &= A \int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + (DX)^T (DX))^{(p+n)/2+\mu}} \\ &= A \int_{\mathbb{R}^{n\times p}} \frac{\det(C)^{-p/2} d[U]}{\det(\mathbb{1}_p + U^T U)^{(p+n)/2+\mu}} \\ &= \underbrace{\left(\frac{1}{\prod_{j=1}^{p} \pi^{n/2}\Gamma[\mu + j/2]/(2\Gamma[\mu + (j+n)/2])}\right)}_{A'} \int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + X^T X)^{(p+n)/2+\mu}} \end{aligned} \tag{2.2}$$

(In the last line we replaced U with X [not the same X as at the start] for ease of keeping notation consistent with the question).

We now wish to deform $\det(\mathbb{1}_p + X^T X)$ into something more manageable. Firstly, we appeal to the following statement:

Let $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{n\times p}$ and $D \in \mathbb{R}^{n\times n}$ be matrices where D is invertible. Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C) \tag{2.3}$$

Using equation (2.3), we see that

$$\begin{aligned}\det(\mathbb{1}_p + X^T X) &= \det(\mathbb{1}_n)\det(\mathbb{1}_p - X^T \mathbb{1}_p^{-1}(-X)) \\ &= \det\begin{pmatrix} \mathbb{1}_p & X^T \\ -X & \mathbb{1}_n \end{pmatrix}\end{aligned}$$

If we then make the substitution $X = (x_1, \tilde{X})$ with $x_1 \in \mathbb{R}^{n\times 1}$ and $\tilde{X} \in \mathbb{R}^{n\times(p-1)}$, then (with $\mathbf{0}_p$ referring to the p-dimensional 0 vector).

$$\begin{aligned}\det\begin{pmatrix} \mathbb{1}_p & X^T \\ -X & \mathbb{1}_n \end{pmatrix} &= \det\begin{pmatrix} \mathbb{1}_p & \begin{pmatrix} x_1^T \\ \tilde{X}^T \end{pmatrix} \\ -(x_1, \tilde{X}) & \mathbb{1}_n \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & \mathbf{0}_{p-1} & x_1^T \\ \mathbf{0}_{p-1} & \mathbb{1}_{p-1} & \tilde{X}^T \\ -x_1 & -\tilde{X} & \mathbb{1}_n \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & \mathbf{0}_{p-1} & x_1^T \\ \mathbf{0}_{p-1} & \mathbb{1}_{p-1} & \tilde{X}^T \\ 0 & -\tilde{X} & \mathbb{1}_n + x_1 x_1^T \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & \mathbf{0}_{p-1} & \mathbf{0}_n \\ \mathbf{0}_{p-1} & \mathbb{1}_{p-1} & \tilde{X}^T \\ \mathbf{0}_n & -\tilde{X} & \mathbb{1}_n + x_1 x_1^T \end{pmatrix} \\ &= \det\begin{pmatrix} \mathbb{1}_{p-1} & \tilde{X}^T \\ -\tilde{X} & \mathbb{1}_n + x_1 x_1^T \end{pmatrix} \qquad (2.4)\end{aligned}$$

Where we performed the elementary row operations (thus preserving the determinant) of $R_3 \mapsto R_3 + x_1 R_1$, followed by $C_3 \mapsto C_3 - x_1^T C_1$.

We can now rescale $\tilde{X}$ with the substitution $\tilde{X} = B\tilde{U}$ where $B = \sqrt{\mathbb{1}_n + x_1 x_1^T}$. Then $d[\tilde{X}] = d[B\tilde{U}] = \det(B)^{(p-1)} d[\tilde{U}]$ - we will calculate $\det(B)$ later. In order to make B more user-friendly, we notice that $\mathbb{1}_n + x_1 x_1^T$ is a real-symmetric positive definite matrix, hence meaning we can decompose it as $\mathbb{1}_n + x_1 x_1^T = QDQ^T$ for a real unitary matrix Q and a diagonal matrix D. This then gives us the following properties:

$$B = \sqrt{\mathbb{1}_n + x_1 x_1^T} = Q\sqrt{D}Q^T \qquad (2.5)$$

$$B^T = (Q\sqrt{D}Q^T)^T = (Q^T)^T \sqrt{D}^T (Q)^T = Q\sqrt{D}Q^T = B \qquad (2.6)$$

$$(B^2)^{-1} = (QDQ^T)^{-1} = QD^{-1}Q^T \qquad (2.7)$$

$$D^k D^m = D^{(k+m)} \quad \forall k, m \in \mathbb{R} \qquad (2.8)$$

Notice then that $B \in \mathrm{Sym}_+(n)$ (retains positive eigenvalues and is symmetric).

After this rescaling, (2.4) then becomes

$$\begin{aligned}
\det\begin{pmatrix} \mathbb{1}_{p-1} & \tilde{X}^T \\ -\tilde{X} & \mathbb{1}_n + x_1x_1^T \end{pmatrix} &= \det\begin{pmatrix} \mathbb{1}_{p-1} & (B\tilde{U})^T \\ -B\tilde{U} & B^2 \end{pmatrix} \\
&= \det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T B^T (B^2)^{-1} B\tilde{U}) \\
&= \det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T (Q\sqrt{D}Q^T)(QD^{-1}Q^T)(Q\sqrt{D}Q^T)\tilde{U}) \\
&= \det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T Q(\sqrt{D}D^{-1}\sqrt{D})Q^T\tilde{U}) \\
&= \det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T Q\mathbb{1}_n Q^T\tilde{U}) \\
&= \det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T\tilde{U})
\end{aligned}$$

We can then calculate $\det(\mathbb{1}_n + x_1x_1^T)$ by factorising into upper-triangular · lower-triangular by performing $C_1 \mapsto C_1 + C_2x_1$.

$$\begin{aligned}
\det(\mathbb{1}_n + x_1x_1^T) &= \det\begin{pmatrix} 1 & x_1^T \\ -x_1 & \mathbb{1}_n \end{pmatrix} \\
&= \det\left[\begin{pmatrix} 1 + x_1^Tx_1 & x_1^T \\ 0 & \mathbb{1}_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x_1 & \mathbb{1}_n \end{pmatrix}\right] \\
&= \det(1 + x_1^Tx_1)\det(\mathbb{1}_n - 0)\det(1)\det(\mathbb{1}_n - 0) \\
&= 1 + x_1^Tx_1
\end{aligned}$$

This then implies that $\det(B) = \det(\sqrt{\mathbb{1}_n + x_1x_1^T}) = (1 + x_1^Tx_1)^{1/2}$ (which is clearly well defined), hence $d[\tilde{X}] = (1 + x_1^Tx_1)^{(p-1)/2}d[\tilde{U}]$.

Combining all of this into our integral in (2.2), we get

$$\begin{aligned}
\int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + X^TX)^{(p+n)/2+\mu}} &= \int_{\mathbb{R}^{n\times p}} d[x_1]d[\tilde{X}]\det\begin{pmatrix} \mathbb{1}_{p-1} & \tilde{X}^T \\ -\tilde{X} & \mathbb{1}_n + x_1x_1^T \end{pmatrix}^{-((p+n)/2+\mu)} \\
&= \int_{\mathbb{R}^{n\times p}} \frac{d[x_1](1 + x_1^Tx_1)^{(p-1)/2}d[\tilde{U}]}{\left[\det(\mathbb{1}_n + x_1x_1^T)\det(\mathbb{1}_{p-1} + \tilde{U}^T\tilde{U})\right]^{(p+n)/2+\mu}} \\
&= \int_{\mathbb{R}^{n\times p}} \frac{d[x_1](1 + x_1^Tx_1)^{(p-1)/2}d[\tilde{U}]}{\left[(1 + x_1^Tx_1)\det(\mathbb{1}_{p-1} + \tilde{U}^T\tilde{U})\right]^{(p+n)/2+\mu}} \\
&= \int_{\mathbb{R}^n} \frac{(1 + x_1^Tx_1)^{(p-1)/2}d[x_1]}{(1 + x_1^Tx_1)^{((p+n)/2+\mu}} \int_{\mathbb{R}^{n\times(p-1)}} \frac{d[\tilde{U}]}{\det(\mathbb{1}_{p-1} + \tilde{U}^T\tilde{U})^{(p+n)/2+\mu}} \\
&= \int_{\mathbb{R}^n} \frac{d[x_1]}{(1 + x_1^Tx_1)^{((n+1)/2+\mu)}} \int_{\mathbb{R}^{n\times(p-1)}} \frac{d[\tilde{U}]}{\det(\mathbb{1}_{p-1} + \tilde{U}^T\tilde{U})^{(p+n)/2+\mu}}
\end{aligned}$$

Each time we perform the iteration, we will get $d[\tilde{X}_j] = (1 + x_j^Tx_j)^{(p-j)/2}d[\tilde{U}]$ over $j = 1, \ldots, p$. Hence, when we see how this behaves on the second last line of the above calculation, we see that this will lead to a factor of $j/2$ in the exponent of the denominator.

Thus, through great toil and hardship, we then arrive at

$$\int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + X^T X)^{(p+n)/2+\mu}} = \prod_{j=1}^{p} \int_{\mathbb{R}^n} \frac{d[x_j]}{(1+x_j^T x_j)^{(n+j)/2+\mu}}$$

Through a change of variables into n-dimensional spherical coordinates where the Jacobian is, disregarding the angular components, r_j^{n-1}, and using a well formulated trick with the Dirac delta function that I wasn't quite able to show, we arrive at

$$\int_{\mathbb{R}^n} \frac{d[x_j]}{(1+x_j^T x_j)^{(n+j)/2+\mu}} = \int_{\mathbb{R}^n} \delta(1 - x^T x) d[x] \int_0^\infty \frac{r_j^{n-1} dr_j}{(1+r_j^2)^{(n+j)/2+\mu}}$$

We then see that the first integral is the volume of the unit sphere which can be computed using the following trick

$$\int_{-\infty}^{\infty} e^{x(ik-a)} dk = 2\pi\delta(x)$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} \delta(1 - x^T x) d[x] &= \int_{\mathbb{R}^n} \left(\int_{-\infty}^{\infty} \exp\left[(1 - x^T x)(it+1)\right] \frac{dt}{2\pi} \right) d[x] \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{(1-(x_1^2+\cdots+x_n^2))(it+1)} dt dx_1 \dots dx_n \\
&= \frac{1}{2\pi} \int_{t=-\infty}^{t=\infty} e^{(it+1)} \left(\int_{-\infty}^{\infty} e^{-(it+1)x^2} dx \right)^n dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it+1} \left(\frac{\pi}{it+1} \right)^{n/2} dt \qquad \text{for } \mathrm{Im}(t) < 1 \\
&= \frac{\pi^{n/2-1}}{2} \int_{-\infty}^{\infty} \frac{1}{(it+1)^{n/2}} e^{it+1} dt
\end{aligned}$$

Then, we can use the fact from Laplace transform theory that

$$\int_0^\infty s^n e^{-(t-a)s} ds = \frac{\Gamma(n+1)}{(t-a)^{n+1}} \quad \Longrightarrow \quad \frac{1}{(it+1)^{n/2}} = \frac{1}{\Gamma(n/2)} \int_0^\infty s^{n/2-1} e^{-(it+1)s} ds$$

to continue the above calculation with

$$\begin{aligned}
&= \frac{\pi^{n/2-1}}{2\Gamma[n/2]} \int_{-\infty}^{\infty} \left(\int_0^\infty s^{n/2-1} e^{-(it+1)s} ds \right) e^{it+1} dt \\
&= \frac{\pi^{n/2-1}}{2\Gamma[n/2]} \int_0^\infty s^{n/2-1} \left(\int_{-\infty}^{\infty} e^{(1-s)(it+1)} dt \right) ds \\
&= \frac{\pi^{n/2-1}}{2\Gamma[n/2]} \int_0^\infty s^{n/2-1} 2\pi\delta(1-s) ds \\
&= \frac{\pi^{n/2}}{\Gamma[n/2]}
\end{aligned}$$

$$\therefore \int_{\mathbb{R}^n} \delta(1 - x^T x) d[x] = \frac{\pi^{n/2}}{\Gamma[n/2]} \tag{2.9}$$

Unfortunately, we notice that the standard definition of the volume of the unit ball is $V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, which is very slightly out from our above calculation, however I was unable to find where this factor went missing. Nonetheless, this $\Gamma[n/2]$ factor cancels out in the final calculation regardless, so it is not too detrimental.

We can then perform the integral in r_j similarly.

$$\begin{aligned}
\int_0^\infty \frac{r_j^{n-1} dr_j}{(1 + r_j^2)^{(n+j)/2+\mu}} &= \int_0^\infty \frac{(r_j^2)^{n/2-1} r_j dr_j}{(1 + r_j^2)^{(n+j)/2+\mu}} \\
&= \frac{1}{2} \int_0^\infty \frac{t^{n/2-1} dt}{(1 + t)^{(n+j)/2+\mu}} \\
&= \frac{1}{2\Gamma[(n+j)/2+\mu]} \int_0^\infty t^{n/2-1} \left(\int_0^\infty s^{((n+j)/2+\mu)-1} e^{-(t+1)s} ds \right) dt \\
&= \frac{1}{2\Gamma[(n+j)/2+\mu]} \int_0^\infty s^{((n+j)/2+\mu)-1} e^{-s} \left(\int_0^\infty t^{n/2-1} e^{-st} dt \right) ds \\
&= \frac{\Gamma[n/2]}{2\Gamma[(n+j)/2+\mu]} \int_0^\infty s^{((n+j)/2+\mu)-1} e^{-s} s^{-n/2} ds \\
&= \frac{\Gamma[n/2]}{2\Gamma[(n+j)/2+\mu]} \int_0^\infty s^{j/2+\mu-1} e^{-s} ds \\
&= \frac{\Gamma[n/2]\Gamma[j/2+\mu]}{2\Gamma[(n+j)/2+\mu]} \qquad (2.10)
\end{aligned}$$

Combining the results from (2.9) and (2.10), we finally see that

$$\begin{aligned}
\int_{\mathbb{R}^{n\times p}} \frac{d[X]}{\det(\mathbb{1}_p + X^T X)^{(p+n)/2+\mu}} &= \prod_{j=1}^{p} \frac{\pi^{n/2}}{\Gamma[n/2]} \frac{\Gamma[n/2]\Gamma[j/2+\mu]}{2\Gamma[(n+j)/2+\mu]} \\
&= \prod_{j=1}^{p} \frac{\pi^{n/2}\Gamma[j/2+\mu]}{2\Gamma[(n+j)/2+\mu]} = \frac{1}{A'}
\end{aligned}$$

Hence, we can conclude that indeed $P(X|C)$ is normalised. $\square$

Part b)

Assume we have measured a time series matrix V with $p \geq n$ and VV^T is invertible, and furthermore assume it follows the distribution given in (2.1). We wish to compute the self-consistent maximal likelihood $P(.|C_0)$ where $C_0 \in \mathcal{C} \in \text{Sym}_+(n)$.

We first wish to calculate the log-likelihood of P, namely

$$\log[P(V|C)] = \frac{p}{2}\log(\det C) - \left(\frac{p+n}{2} + \mu\right)\log(\det(\mathbb{1}_p + V^TCV)) - \sum_{j=1}^{p}\log\left(\frac{\pi^{n/2}\Gamma[\mu + j/2]}{2\Gamma[\mu + (j+n)/2]}\right)$$
$$= \frac{p}{2}\operatorname{tr}\log(C) - \left(\frac{p+n}{2} + \mu\right)\operatorname{tr}\log(\mathbb{1}_p + V^TCV) - \sum_{j=1}^{p}\log\left(\frac{\pi^{n/2}\Gamma[\mu + j/2]}{2\Gamma[\mu + (j+n)/2]}\right)$$

To find where this is maximised, i.e. at the extremum C_0, we wish to solve $\nabla_C \log[P(V|C)] = 0$. We will make use of the fact that in lectures we calculated $\nabla_A \operatorname{tr}\log(A) = A^{-1}$ for $\mathcal{O} = \text{Sym}(n) \supset \text{Sym}_+(n)$. We also note that
$\nabla_A(\mathbb{1}_p + V^TCV) = V^T(.)V$ (poorly defined notation sorry).

$$\nabla_C \log[P(V|C)] = \frac{p}{2}C^{-1} - \left(\frac{p+n+2\mu}{2}\right)V(\mathbb{1}_p + V^TCV)^{-1}V^T = 0$$

Which tells us that C_0, the extremum, must satisfy

$$\frac{p}{2}C_0^{-1} - \left(\frac{p+n+2\mu}{2}\right)V(\mathbb{1}_p + V^TC_0V)^{-1}V^T = 0$$

We can then appeal to the Neumann series for operators that states
$(\mathbb{1} - A)^{-1} = \sum_{j=0}^{\infty} A^j$. Plugging this in to our above equation, we get

$$\frac{p}{2}C_0^{-1} - \left(\frac{p+n+2\mu}{2}\right)\sum_{k=0}^{\infty}(-1)^k V(V^TC_0V)^kV^T = 0$$
$$\implies \frac{p}{p+n+2\mu}C_0^{-1} = \sum_{k=0}^{\infty}(-1)^k(VV^TC_0)^{k+1}C_0^{-1}$$
$$\implies \frac{p}{p+n+2\mu}\mathbb{1}_n = (VV^TC_0)\sum_{k=0}^{\infty}(-1)^k(VV^TC_0)^k$$
$$\implies \frac{p}{p+n+2\mu}\mathbb{1}_n = (VV^TC_0)(\mathbb{1}_n + VV^TC_0)^{-1}$$
$$\implies \frac{p}{p+n+2\mu}(\mathbb{1}_n + VV^TC_0) = VV^TC_0$$
$$\therefore C_0 = \frac{p}{n+2\mu}(VV^T)^{-1}$$

We know that our quantity for C_0 is well defined because VV^T is invertible by assumption. Hence, C_0 is the maximal likelihood value for the parameter C for the correlated real Cauchy-Lorentz ensemble. □

Q3. Complex Uncorrelated Wishart-Laguerre Ensemble

Let $X \in \mathbb{C}^{n\times p}$ be drawn from the χGUE with distribution

$$P(X) = \frac{1}{(\pi/n)^{np}} e^{-n \operatorname{tr} XX^\dagger}$$

We wish to show that the limiting level density yields the Marčenko-Pastur distribution for $n, p \to \infty$ with $\lim_{n\to\infty} p/n = \gamma \in [1, \infty)$.

We start with the following equation

$$\int_{\mathbb{C}^{n\times p}} d[X] \partial_{X_{ab}} \left(\{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{cd} X_{ef} e^{-n \operatorname{tr} XX^\dagger} \right) = 0 \tag{3.1}$$

Where $a, c, d, e = 1, \dots, n$ and $b, f = 1, \dots, p$. We can then explicitly calculate the matrix derivatives of X and $X^\dagger$

$$\partial_{X_{ab}} X = E_{ab} \qquad\qquad \partial_{X_{ab}} X^\dagger = 0$$

Where $E_{ab} \in \mathbb{C}^{n\times p}$ is the matrix with a 1 at the (a, b) entry. The second equation follows from the fact that $\frac{d}{dz}(\bar{z}) = 0$. Then we can write

$$\begin{aligned}
\partial_{X_{ab}} X_{ef} e^{-n \operatorname{tr} XX^\dagger} &= (\partial_{X_{ab}} X_{ef}) e^{-n \operatorname{tr} XX^\dagger} + X_{ef} e^{-n \operatorname{tr} XX^\dagger} \partial_{X_{ab}} \operatorname{tr}(-nXX^\dagger) \\
&= \delta_{ae}\delta_{bf} e^{-n \operatorname{tr} XX^\dagger} - nX_{ef} e^{-n \operatorname{tr} XX^\dagger} \operatorname{tr}(E_{ab} X^\dagger) \\
&= -(nX^*_{ab} X_{ef} - \delta_{ae}\delta_{bf}) e^{-n \operatorname{tr} XX^\dagger}
\end{aligned}$$

Using the resolvent formula from the notes, we can also write

$$\begin{aligned}
\partial_{X_{ab}} [(z\mathbb{1}_n - XX^\dagger)^{-1}] &= -(z\mathbb{1}_n - XX^\dagger)^{-1} [\partial_{X_{ab}} (z\mathbb{1}_n - XX^\dagger)] (z\mathbb{1}_n - XX^\dagger)^{-1} \\
&= (z\mathbb{1}_n - XX^\dagger)^{-1} E_{ab} X^\dagger (z\mathbb{1}_n - XX^\dagger)^{-1}
\end{aligned}$$

Which gives us

$$\partial_{X_{ab}} \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{cd} = \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{ca} \{X^\dagger (z\mathbb{1}_n - XX^\dagger)^{-1}\}_{bd}$$

We then return to (3.1) and can now write

$$\begin{aligned}
&\int_{\mathbb{C}^{n\times p}} d[X] \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{cd} (nX^*_{ab} X_{ef} - \delta_{ae}\delta_{bf}) e^{-n \operatorname{tr} XX^\dagger} \\
&= \int_{\mathbb{C}^{n\times p}} d[X] \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{ca} \{X^\dagger (z\mathbb{1}_n - XX^\dagger)^{-1}\}_{bd} X_{ef} e^{-n \operatorname{tr} XX^\dagger}
\end{aligned}$$

We then use the fabled loop equations to contract our indices. We first set $a = c$, $d = e$ and $b = f$, so summing over the indices we see that the relevant part of the first integral becomes

$$\begin{aligned}
&\sum_{a=1}^{n} \sum_{b=1}^{p} \sum_{d=1}^{n} \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{ad} (nX^*_{ab} X^T_{bd} - \delta_{ad}\delta_{bb}) \\
&= \sum_{a=1}^{n} \sum_{d=1}^{n} \{(z\mathbb{1}_n - XX^\dagger)^{-1}\}_{ad} (n(X^* X^T)_{ad} - p\delta_{ad}) \\
&= n \operatorname{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} (X^* X^T)^T - p \operatorname{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} \\
&= n \operatorname{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} XX^\dagger - p \operatorname{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}
\end{aligned}$$

Performing a similar operation for the right hand side, we then arrive at the first loop equation (where the factor of $(\pi/n)^{np}$ cancels out on both sides of our equation)

$$\begin{aligned}\langle n\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}XX^\dagger - p\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\rangle \\ = \langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\,\mathrm{tr}\,X^\dagger(z\mathbb{1}_n - XX^\dagger)^{-1}X\rangle \end{aligned} \tag{3.2}$$

We then appeal to the fact that

$$(z\mathbb{1}_n - XX^\dagger)^{-1}XX^\dagger = z(z\mathbb{1}_n - XX^\dagger)^{-1} - \mathbb{1}_n$$

Which simplifies (3.2) to

$$\begin{aligned} n\langle \mathrm{tr}(z(z\mathbb{1}_n - XX^\dagger)^{-1} - \mathbb{1}_n)\rangle - p\langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\rangle \\ = \langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1})\,\mathrm{tr}((z(z\mathbb{1}_n - XX^\dagger)^{-1} - \mathbb{1}_n))\rangle \end{aligned}$$

$$\begin{aligned} \implies nz\langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\rangle - n^2 - p\langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\rangle \\ = \langle \big(\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\big)^2\rangle - n\langle \mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\rangle \end{aligned}$$

Dividing by n^2 and appropriately rearranging gives us

$$z\left\langle\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\right)^2\right\rangle + \left(\frac{p}{n} - 1 - z\right)\left\langle\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1}\right\rangle + 1 = 0 \tag{3.3}$$

We then make the following assumption for the limiting Green function $G(z)$ of the random matrix $XX^\dagger$

$$\lim_{n,p\to\infty}\left\langle\left|\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right|^2\right\rangle = 0$$

We can then simplify (3.3) with the following expansion (apologies for the poor layout - still getting the hang of LaTeX)

$$\begin{aligned} & z\left\langle\left(\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right) + G(z)\right)^2\right\rangle \\ & + \left(\frac{p}{n} - 1 - z\right)\left\langle\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right) + G(z)\right\rangle + 1 = 0 \\ \implies & z\left\langle\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right)^2\right\rangle + 2z\left\langle\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right)G(z)\right\rangle + z\left\langle G(z)^2\right\rangle \\ & + \left(\frac{p}{n} - 1 - z\right)\left\langle\left(\frac{1}{n}\,\mathrm{tr}(z\mathbb{1}_n - XX^\dagger)^{-1} - G(z)\right)\right\rangle + \left(\frac{p}{n} - 1 - z\right)\langle G(z)\rangle + 1 = 0 \end{aligned}$$

If we then take the limit $\lim_{n,p\to\infty}$, remembering that $\gamma = \lim_{n,p\to\infty} p/n$, we arrive at the quadratic equation in $G(z)$

$$zG^2(z) + (\gamma - 1 - z)G(z) + 1 = 0$$

Which yields us the solutions

$$G(z) = \frac{z + 1 - \gamma \pm \sqrt{(z + 1 - \gamma)^2 - 4z}}{2z}$$

But since we need the asymptotic behaviour of $G(z) \approx 1/z$ for $|z| \to \infty$, we will take the negative sign. Hence, after factorising the inside of the square root, we see that

$$G(z) = \frac{z + 1 - \gamma - \sqrt{[z - (\sqrt{\gamma} + 1)^2][z - (\sqrt{\gamma} - 1)^2]}}{2z}$$

For a real z, we require an imaginary part of $G(z)$ in order for $\hat{\rho}(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G(\lambda - i\varepsilon)$ to be well defined. This gives us the support $z \in ((\sqrt{\gamma} - 1)^2, (\sqrt{\gamma} + 1)^2)$. Hence,

$$\begin{aligned} \hat{\rho}(\lambda) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G(\lambda - i\varepsilon) \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} \frac{\lambda - i\varepsilon + 1 - \gamma - \sqrt{[\lambda - i\varepsilon - (\sqrt{\gamma} + 1)^2][\lambda - i\varepsilon - (\sqrt{\gamma} - 1)^2]}}{2\lambda - i\varepsilon} \end{aligned}$$

$$\therefore \quad \hat{\rho}(\lambda) = \frac{\sqrt{[(\sqrt{\gamma} + 1)^2 - \lambda][\lambda - (\sqrt{\gamma} - 1)^2]}}{2\pi\lambda}$$

Therefore we see that we can write λ_+ and λ_- as

$$\lambda_+ = (\sqrt{\gamma} + 1)^2 \qquad \lambda_- = (\sqrt{\gamma} - 1)^2$$

Hence we have shown that the limiting level density for the χGUE yields the Marčenko-Pastur distribution as desired. □

Q4. Level spacing distribution for GSE matrix H

Let H be a Hermitian self-dual matrix, as defined in Ex 2.1 to satisfy $H = Q + Q^\dagger$, with dimension $N = 2$. Let $X_1 \in \mathbb{C}^{2\times 2}$ and $X_2, X_3 \in \mathrm{ASym}_{\mathbb{C}}(2)$, and let $a, b, c, d, e, f \in \mathbb{C}$ and $a', d' \in \mathbb{R}$. Then all of these constraints give

$$Q = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_1^T \end{pmatrix} \qquad X_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$$

$$\begin{aligned} \implies H = Q + Q^\dagger &= \begin{pmatrix} X_1 + X_1^\dagger & X_2 + X_3^\dagger \\ X_3 + X_2^\dagger & (X_1 + X_1^\dagger)^T \end{pmatrix} \\ &= \begin{pmatrix} a' & b + \bar{c} & 0 & e - \bar{f} \\ \overline{b + \bar{c}} & d' & -(e - \bar{f}) & 0 \\ 0 & \overline{-e - \bar{f}} & a' & \overline{b + \bar{c}} \\ \overline{e - \bar{f}} & 0 & b + \bar{c} & d' \end{pmatrix} \end{aligned}$$

With suitable relabelling for $a, d \in \mathbb{R}$ and $b, c \in \mathbb{C}$ we get

$$H = \begin{pmatrix} a & b & 0 & c \\ \bar{b} & d & -c & 0 \\ 0 & -\bar{c} & a & \bar{b} \\ \bar{c} & 0 & b & d \end{pmatrix}$$

We can now calculate the two doubly degenerate eigenvalues of H

$$\begin{aligned} \det(H - \lambda \mathbb{1}_4) &= \det \begin{pmatrix} a-\lambda & b & 0 & c \\ \bar{b} & d-\lambda & -c & 0 \\ 0 & -\bar{c} & a-\lambda & \bar{b} \\ \bar{c} & 0 & b & d-\lambda \end{pmatrix} \\ &= (a-\lambda)\det\begin{pmatrix} d-\lambda & -c & 0 \\ -\bar{c} & a-\lambda & \bar{b} \\ 0 & b & d-\lambda \end{pmatrix} - b \det \begin{pmatrix} \bar{b} & -c & 0 \\ 0 & a-\lambda & \bar{b} \\ \bar{c} & b & d-\lambda \end{pmatrix} \\ &\quad - c \det \begin{pmatrix} \bar{b} & d-\lambda & -c \\ 0 & -\bar{c} & a-\lambda \\ \bar{c} & 0 & b \end{pmatrix} \\ &= (a-\lambda)(d-\lambda)\left((a-\lambda)(d-\lambda) - b\bar{b} - c\bar{c}\right) - b\bar{b}\left((a-\lambda)(d-\lambda) - b\bar{b} - c\bar{c}\right) \\ &\quad - c\bar{c}\left((a-\lambda)(d-\lambda) - b\bar{b} - c\bar{c}\right) \\ &= \left((a-\lambda)(d-\lambda) - (|b|^2 + |c|^2)\right)^2 \\ &= \left(\lambda^2 - (a+d)\lambda + (ad - (|b|^2 + |c|^2))\right)^2 \end{aligned}$$

Solving the quadratic inside of the square (where the square is what gives us the double degeneracy that we proved in Q1) yields us

$$\begin{aligned} \lambda_\pm &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - (|b|^2 + |c|^2))}}{2} \\ &= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4(|b|^2 + |c|^2)}}{2} \end{aligned}$$

Hence, we can write their difference

$$\Delta\lambda = \sqrt{(a-d)^2 + 4(bb^* + cc^*)}$$

To calculate the level spacing distribution, we wish to calculate

$$p_{\text{sp}}(s,\mathcal{I}) = \frac{\left\langle \sum_{E_j \in \mathcal{I}\setminus\{E_N\}} \delta(s - (E_{j+1} - E_j)/\bar{s}) \right\rangle}{\left\langle \sum_{E_j \in \mathcal{I}\setminus\{E_N\}} 1 \right\rangle} = \langle \delta(s - \Delta\lambda/\bar{s}) \rangle$$

Where we use $\mathcal{I} = \mathbb{R}$, hence simplifying the equation. Our first substitution will be to simplify the bb^* and cc^* terms.

$$\begin{array}{cccc} a' = a & d' = d & \text{Re}(b) = b'\cos(\theta_b) & \text{Im}(b) = b'\sin(\theta_b) \\ a' \in (-\infty,\infty) & d' \in (-\infty,\infty) & b' \in (0,\infty) & \theta_b \in [0,2\pi] \end{array}$$

$$\begin{array}{cc} \text{Re}(c) = c'\cos(\theta_c) & \text{Im}(c) = c'\sin(\theta_c) \\ c' \in (0,\infty) & \theta_c \in [0,2\pi] \end{array}$$

$$\implies \Delta\lambda = \sqrt{(a'-d')^2 + 4b'^2 + 4c'^2}$$

$$d[H] = [da][dd][d\text{Re}(b)][d\text{Im}(b)][d\text{Re}(c)][d\text{Im}(c)] = b'c'\ [da'][dd'][db'][dc'][d\theta_b][d\theta_c]$$

We then make the better substitutions for $w, x, y, z \in \mathbb{R}$ where the support remains the same

$$\begin{array}{cccc} w = a' + d' & x = a' - d' & y = 2b' & z = 2c' \\ w \in (-\infty,\infty) & x \in (-\infty,\infty) & y \in (0,\infty) & z \in (0,\infty) \end{array}$$

$$\implies \Delta\lambda = \sqrt{x^2 + y^2 + z^2} \qquad d[H] = \frac{1}{32}\, y\, z\, dw dx dy dz d\theta_b d\theta_c$$

We can then make an even better substitution into spherical coordinates, where we note in particular that the domain of the azimuthal angle is $\psi \in [0, \pi/2]$ because we have $b', c' \in (0,\infty)$ instead of $(-\infty,\infty)$.

$$w = \bar{S} \qquad x = r\cos(\theta) \qquad y = r\sin(\theta)\sin(\psi) \qquad z = r\sin(\theta)\cos(\psi)$$

$$\implies d[H] = \frac{1}{32}(r^2\sin\theta)(r^2\sin^2\theta\sin\psi\cos\psi)d\bar{S}drd\theta d\psi d\theta_b d\theta_c$$

$$\Delta\lambda = r$$

$$\begin{array}{cc} r \in (0,\infty) & w \in (-\infty,\infty) \\ \theta \in [0,\pi] & \psi \in [0,\pi/2] \\ \theta_b \in [0,2\pi] & \theta_c \in [0,2\pi] \end{array}$$

Hence we can write (where C is the normalising constant for the GSE)

$$\begin{aligned}
p_{\mathrm{sp}}(s) &= \langle \delta(s - \Delta\lambda/\bar{s}) \rangle \\
&= \frac{1}{C} \int_{\mathbb{H}^2} e^{-n \operatorname{tr} H^2} \delta(s - \Delta\lambda/\bar{s}) d[H] \\
&= \frac{1}{C} \int_{\mathbb{H}^2} e^{-4(2a^2+2d^2+4bb^*+4cc^*)} \delta(s - \Delta\lambda/\bar{s}) d[H] \\
&= \frac{1}{C} \int_{D} e^{-4(w^2+x^2+y^2+z^2)} \delta(s - \Delta\lambda/\bar{s}) d[H] \\
&= \frac{1}{32C} \int_{D} r^4 \sin^3\theta \sin\psi \cos\psi e^{-4(\bar{S}^2+r^2)} \delta(s - r/\bar{s}) d\bar{S} dr d\theta d\psi d\theta_b d\theta_c \\
&= \frac{1}{32C} \int_0^{2\pi} \int_0^{2\pi} d\theta_b d\theta_c \int_{-\infty}^{\infty} e^{-4\bar{S}^2} d\bar{S} \int_0^{\pi/2} \sin\psi \cos\psi d\psi \int_0^{\pi} \sin^3\theta d\theta \int_0^{\infty} r^4 e^{-4r^2} \delta(s - r/\bar{s}) dr \\
&= \frac{1}{32C} (2\pi)^2 \left(\frac{\sqrt{\pi}}{2}\right) \left(\frac{1}{2}\right) \left(\frac{4}{3}\right) \left(\bar{s}^5 s^4 e^{-4\bar{s}^2 s^2}\right) \\
&= D \bar{s}^5 s^4 e^{-4\bar{s}^2 s^2}
\end{aligned}$$

We then use the two facts that

$$\int_{-0}^{\infty} p_{\mathrm{sp}}(s) ds = 1 \qquad\qquad \langle S \rangle = \int_0^{\infty} s p_{\mathrm{sp}}(s) ds = 1$$

$$\implies D\bar{s}^5 \frac{3\sqrt{\pi}}{256\bar{s}^5} = 1 \qquad\qquad \implies D\bar{s}^5 \frac{1}{64\bar{s}^6} = 1$$

$$\therefore D = \frac{2^6}{\Gamma(5/2)} \qquad\qquad \implies \bar{s} = \frac{1}{\Gamma(5/2)}$$

Combining all of this together, and noticing that $\Gamma(3) = 2$, especially noticing the factor of 4 in the exponent, we arrive at the beautiful equation of the level spacing distribution for the 4×4 GSE matrix H, namely

$$p_{\mathrm{sp}}(s) = 2 \frac{\Gamma(3)^5}{\Gamma(5/2)^6} s^4 \exp\left[-\left(\frac{\Gamma(3)}{\Gamma(5/2)} \right)^2 s^2 \right]$$

□